A quick introduction to continuous groups

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An introduction to continuous groups is presented.

I. INTRODUCTION

The most common groups you encounter as a nuclear/particle researcher will be what are known as "continuous groups". Of these, the groups $SU(N)$ and $SO(N)$ play an especially important role in particle physics.

This document is not intended to offer a complete, or mathematically rigorous, treatment of the subject. Instead, the intention of this document is to introduce continuous groups to students of particle physics so that they will gain a level of comfort with the subject that they can then utilize in furthering their understanding with future study.

II. DEFINITION OF A GROUP

To define what a group is I will use the two-dimensional rotation group, $SO(2)$, with group elements R_θ as an example, where

$$R_\theta \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

We say that elements g_i form a group if they obey the following

1. Closure: For any group members g_i, and g_j, then $g_i \cdot g_j$ is also a member of the group.

   In our $SO(2)$ example we have $R_\alpha \cdot R_\beta = R_{\alpha+\beta}$, which is clearly within the group.

2. Associativity: $g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$

   In our $SO(2)$ example we have $R_\alpha \cdot (R_\beta \cdot R_\gamma) = (R_\alpha \cdot R_\beta) \cdot R_\gamma$. And from our experience with R_θ elements, we know that associativity holds.

3. Identity: There exists an identity element I such that $g_i \cdot I = I \cdot g_i = g_i$

   In our $SO(2)$ example we have $I = R_{\theta=0}$.

4. Inverse: For every group element g_i there exists an element g_i^{-1} within the group such that $g_i \cdot g_i^{-1} = I$.

   In our $SO(2)$ example we have $R_\theta^{-1} = R_{-\theta}$.

III. DEFINITIONS OF THE GROUPS $SU(N)$ AND $SO(N)$

Special unitary matrices of dimension N are the group of all N by N unitary matrices (i.e. $U^\dagger U = 1$), represented here as U, that have $det(U) = 1$ (special) This group is denoted as $SU(N)$.

Similarly, the group of all N by N orthogonal matrices (i.e. $O^T O = 1$), represented here as O, that have $det(O) = 1$ are called the group of special orthogonal matrices of dimension N. This group is denoted as $SO(N)$. In short:

- $S \rightarrow$ unit determinate
- $O \rightarrow$ orthogonal matrix
- $U \rightarrow$ unitary matrix
- $N \rightarrow$ dimension of the group.

What do these groups do? The group members of $SO(N)$ can be used to transform N-component real vectors, while group members of $SU(N)$ are used to transform N-component complex vectors. It is important to note that both of these transformations will maintain an invariant length (i.e. $\vec{v} \cdot \vec{v} =$ constant, where $\vec{v} \cdot \vec{v} \equiv \vec{v}^\dagger \vec{v}$ for the complex case).

IV. THE GROUPS $U(1)$, $SO(2)$, $SU(2)$, $SO(3)$, AND $SU(3)$

A. $U(1)$

One of the simplest groups that can be studied is the $U(1)$ group. This group is comprised by all unitary matrices of dimension 1. Since this group is dimension 1, we do not need to worry about the determinate. For this reason, we do not talk of $SU(1)$, instead we will only consider $U(1)$ and find that the unitarity constraint "fixes" the overall size of these complex numbers.

We start out by letting U be a group member of $U(1)$. Furthermore, since U is just some complex number, we can set $U = re^{i\theta}$, with r and θ being free real parameters that are consistent with the unitarity requirement $U^\dagger U = 1$. When employing the unitary requirement it is easy to see that $r^2 = 1$, and since $r \in$ the group of purely real numbers $\Re$, then $r = \pm 1$. Thus, we take

$$U = e^{i\theta}.$$

What does the $U(1)$ group do? A $U(1)$ group member simply takes a complex number and changes the phase of that number.

The group $U(1)$ is the gauge group that is responsible for electrodynamics. It is beyond the scope of this document to fully describe what is meant by a "gauge symmetry", however, I will briefly sketch the highlights: When the Lagrangian for a spin-1/2 Dirac-field is demanded to be invariant under a local phase transformation of that Dirac-field ($\psi \to e^{i\theta(x)}\psi$, where $\theta(x)$ means θ is position dependent), extra terms must be introduced. These extra terms are identical to the terms one would include if the original Lagrangian included, in addition to the spin-half Dirac field, the photon field (and the interaction between the two). For this reason, it is sometimes said that electrodynamics is a result of $U(1)$ gauge symmetry.

This would be a mere curiosity if it were not for the fact that the weak and strong forces can be similarly generated by gauge groups. The strong force is generated by a $SU(3)$ gauge symmetry and the weak force is generated by a $SU(2)$ gauge symmetry. This is why you will sometimes see the entire physics of these forces in the Standard Model described simply as

$$SU(3) \times SU(2) \times U(1).$$

B. $SO(2)$

The most intuitive group for us to study is $SO(2)$. Undoubtedly, you have worked with this group many many times, but did not know the group theory terminology for it. Because of your familiarity with calculating quantities using this group, this group provides a great opportunity to build up your knowledge of group theory terminology.

The group $SO(2)$ is the group of all 2×2 orthogonal matrices that have unit determinate. The orthogonality requirement is defined as $O^T O = 1$, where O^T is the transpose of O (i.e. for each $o_{i,j}$ matrix element of O, $(o_{i,j})^T = o_{j,i}$).

Since this group is so familiar and is of low dimension, it will not be too bothersome to present the details of how one might derive the group members of $SO(2)$ in two different ways. In the first method we will derive the group members by directly calculating the individual matrix elements of O. In the second method we will derive the group members by finding the "generator" of the group $SO(2)$.

In the first method we want to calculate the matrix elements of O. If we write the matrix O as

$$O = \begin{bmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{bmatrix}$$

then the condition $O^T O = 1$ reads

$$O^T O = \begin{bmatrix} o_{11} & o_{21} \\ o_{12} & o_{22} \end{bmatrix} \begin{bmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This implies that

$$o_{11}^2 + o_{21}^2 = 1 \quad (1)$$
$$o_{11}o_{12} + o_{21}o_{22} = 0 \quad (2)$$
$$o_{12}^2 + o_{22}^2 = 1. \quad (3)$$

While the condition $det(O) = 1$ gives

$$o_{11}o_{22} - o_{12}o_{21} = 1. \quad (4)$$

Eqns. 1 and 3 tells us that $-1 \leq o_{11}, o_{12}, o_{21}, o_{22}, \leq +1$. This means that we can parameterize one of these matrix elements by a function that spans from -1 to +1. In particular, we can choose $o_{11} \equiv \cos\theta$, where θ is a free parameter (that may end up being restricted in value when satisfying Eqns. 1 through 4). After performing some rudimentary algebra we have

$$O = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

and we see there was no need to make any restrictions on the θ parameter. We can now identify the group members of $SO(2)$ as the rotation matrices for two-dimensional space.

Now, we will derive the group members of $SO(2)$ a second way. This derivation is easier but will use some more advanced matrix algebra. I'll try to be careful about introducing the matrix algebra. First let me define a matrix A and the matrix e^A. The matrix e^A is defined as follows,

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

We will need a couple more matrix identities:

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ (AB)^T &= B^T A^T \end{aligned}$$

This means that

$$(e^A)^T = \sum_{j=0}^{\infty} \frac{(A^j)^T}{j!} = \sum_{j=0}^{\infty} \frac{(A^T)^j}{j!} = e^{A^T}$$

The orthogonality condition can now be written as:

$$O^T O = (e^A)^T e^A = e^{A^T} e^A$$

At this point is is tempting to write $e^{A^T} e^A = e^{(A^T + A)}$, but we need to be careful! In general, $e^A e^B = e^{A+B}$, ONLY IF $[A, B] = 0$. For this case we are in luck, $[A^T, A] = 0$.

We can now write the constraint

$$e^{(A^T + A)} = 1 \Rightarrow A^T + A = 0$$

Therefore A is anti-symmetric. This means that we can write A as

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \theta,$$

where θ is a free real parameter. Also, we can now clearly see that $[A^T, A] = 0$.

At this point we should check that e^A is equal to the rotation matrix we obtained earlier. Also, this is a good opportunity to introduce a new group theory term: generator. Let us define J such that

$$A = iJ\theta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \theta.$$

This J is called the generator of the $SO(2)$ group and is given by

$$J = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Now lets see how $e^{iJ\theta}$ is equal to the 2-dimensional rotational matrix:

$$O = e^{iJ\theta} = I + i\frac{J\theta}{1!} - \frac{(J\theta)^2}{2!} - i\frac{(J\theta)^3}{3!} + \frac{(J\theta)^4}{4!} \cdots$$

We notice that

$$J^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Which allows us to write

$$\begin{aligned} O = e^{iJ\theta} &= I + iJ\frac{\theta}{1!} - \frac{\theta^2}{2!} - iJ\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \cdots \\ &= I\left\{1 - \frac{\theta^2}{2!} + \cdots\right\} + iJ\left\{\frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots\right\} \\ &= I\cos\theta + iJ\sin\theta \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \end{aligned}$$

as we saw above.

We have now seen two ways to calculate the group members of $SO(2)$: By direct calculation, and by determination and use of the group generator.

C. A relationship between $U(1)$ and $SO(2)$

What about the generator for $U(1)$? Well, it is simply equal to 1. One interesting detail regarding the generators for $SO(2)$ and $U(1)$ is that there is exactly one generator for each of these continuous groups. Knowing this, we might expect there to be some connection between these two groups. There is, in fact, a connection.

Let me define a complex vector V with a real x-component V_x and a real y-component V_y such that $V = V_x + iV_y$. What does a $U(1)$ transformation $V \to V'$ look like?

$$\begin{aligned} V' &= e^{-i\theta}(V_x + iV_y) \\ &= (V_x\cos\theta + V_y\sin\theta) + \\ &\quad i(V_y\cos\theta - V_x\sin\theta) \end{aligned}$$

This implies that

$$\begin{aligned} V'_x &= V_x\cos\theta + V_y\sin\theta \\ V'_y &= V_y\cos\theta - V_x\sin\theta. \end{aligned}$$

We obtain a 2-dimensional rotation using the $U(1)$ group when we define a vector as $V = V_x + iV_y$. This example nicely illustrates the connection between the $U(1)$ and $SO(2)$ groups ☺.

D. $SU(2)$

$SU(2)$ is another group with which you have some experience. We will investigate this group by looking at the generators of the group. One nice feature of this group is that we will have three separate generators that will allow us to talk about the concept of a Lie algebra.

Let U be a group member of $SU(2)$ and define the matrix A such that $U = e^A$. In what follows we will need the useful matrix identity:

$$det(e^A) = e^{tr(A)},$$

where the trace of A is the sum of the diagonal elements (i.e. $tr(A) = \sum_i A_{i,i}$). Now, if we set an arbitrary group object U of $SU(2)$ as $U = e^A$, then the requirement that $det(U) = 1$ becomes

$$det(U) = det(e^A) = e^{tr(A)} = 1.$$

This implies that $tr(A) = 0$.

Note: Since, in the previous subsection, the A matrix of $SO(2)$ was antisymmetric, we did not have to worry about $tr(A) = 0$. An antisymmetric matrix automatically has diagonal elements all equal to zero.

The unitarity condition $U^\dagger U = 1$ implies that $A^\dagger + A = 0$ (i.e. A is anti-hermitian). The condition $tr(A) = 0$ gives us

$$a_{11} + a_{22} = 0. \tag{5}$$

That A is anti-hermitian tells us that

$$a^*_{11} + a_{11} = 0 \tag{6}$$
$$a^*_{21} + a_{12} = 0 \tag{7}$$
$$a^*_{22} + a_{22} = 0. \tag{8}$$

From equation 6 and 8 we can see that a_{11} and a_{22} are completely imaginary. The other two elements a_{12} and a_{21} can have both real and imaginary parts. From this we can see that there are six independent quantities. In addition to a_{11} and a_{22} being imaginary, Eqn 5 tells us that $a_{11} = -a_{22}$. This reduces the number of independent elements to five. The last equation at our disposal, Eqn. 7, tells us that the real part of a_{21} equals the real part of $-a_{12}$, and that the imaginary part of a_{21} is equal to the imaginary part of a_{12}. We have exhausted all of our equations (Eqns 5 through 8), and have three independent matrix elements. Summarizing we have:

$$Im(a_{12}) = Im(a_{21}) \tag{9}$$
$$-Re(a_{12}) = Re(a_{21}) \tag{10}$$
$$-Im(a_{11}) = Im(a_{22}) \tag{11}$$
$$Re(a_{11}) = Re(a_{22}) = 0. \tag{12}$$

We have a matrix A with three independent matrix elements. This means that we can construct the matrix A as a linear combination of three independent matrices. Let me define three such independent matrices as A_1, A_2, and A_3. A convenient choice is:

$$A_1 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, A_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A_3 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

where we have utilized Eqn. 9 to construct A_1, Eqn. 10 to construct A_2, and Eqns. 11 and 12 to construct A_3. Note: The factor $\frac{1}{2}$ is not absolutely necessary, but introduced for later convenience. In fact, there are an infinite number of ways one could define A_1, A_2, and A_3.

The linear combination can be written

$$A = A_1\theta_1 + A_2\theta_2 + A_3\theta_3,$$

where the θ_i are independent parameters.

If we write $e^A = e^{i\vec{S}\cdot\vec{\theta}}$, then we find that the generators of the $SU(2)$ group are the spin matrices in the Pauli basis:

$$S_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, S_2 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, S_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This means that the $SU(2)$ group is the group that deals with spin. It should be noted that at no time did we use quantum mechanics. The mathematics of spin in quantum mechanics is a natural consequence of group theory!

We can now introduce the concept of Lie algebra. The Lie algebra is simply the commutation relations between the generators of the group. For $SU(2)$ we have as the Lie algebra

$$[S_i, S_j] = i\epsilon_{ijk}S_k. \tag{13}$$

The constants ϵ_{ijk} are called the "structure constants."

Previously, I mentioned that the electromagnetic field could be seen as a consequence of $U(1)$ gauge symmetry and that the weak interaction was a consequence of $SU(2)$ gauge symmetry. Now that we have seen that there is one generator for the $U(1)$ group and three generators for $SU(2)$, it is natural to ask how the number of generators is related to the produced fields. It turns out that for each generator there is a field. In the case of $SU(N)$, there are $N^2 - 1$ generators. For electromagnetism ($U(1)$) we have one generator and one field (photon). For the weak interaction ($SU(2)$) we have three generators and three fields (W^+, W^-, and Z^0). For the $SU(3)$ gauge-field (QCD, or the strong interaction) there are eight generators and eight colored gluon fields.

E. $SO(3)$

As you probably have suspected, $SO(3)$ is the group that is responsible for rotations in three dimensional space. Because you already have worked with this group many times and are comfortable with it, I will not attempt to construct the group members but merely state a few results.

If we write the group member O of $SO(3)$ as $e^{i\vec{J}\vec{\theta}}$, the generators of the $SO(3)$ group can be written as

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, J_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Lie algebra for $SO(3)$ is

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{14}$$

We also know that $\vec{B}\times\vec{C}$ transforms as a vector under an $SO(3)$ transformation. Because of this, we have grown accustomed to writing quantities like $\vec{L} = \vec{r}\times\vec{p}$. A different way to write this is $L_k = \sum_{i,j}\epsilon_{ijk}\, r_i\, p_j$ (or simply $L_k = \epsilon_{ijk}\, r_i\, p_j$, if we suppress the summation notation).

NOTE: Being able to write antisymmetric combinations of vector quantities $\epsilon_{ijk}\, A_i\, B_j$, that also transform as vectors, will be very useful when we talk about multiplets under $SU(3)$.

F. A relationship between $SU(2)$ and $SO(3)$

The case for there to be a relationship between $SU(2)$ and $SO(3)$ is very strong. We can see from Eqns. 13 and 14 that $SU(2)$ and $SO(3)$ share the same Lie algebra! To clearly see the connection between the two groups, let me define the vectors $\vec{V}$ and $\vec{V'}$ as

$$\vec{V} \equiv V_x\sigma_x + V_y\sigma_y + V_z\sigma_z, \tag{15}$$
$$\vec{V'} \equiv e^{i\vec{S}\theta}Ve^{-i\vec{S}\theta} \tag{16}$$

To make life easier, I will only consider rotations about the z-axis:

$$\vec{V'} = \left[cos(\frac{\theta}{2}) + i\sigma_z sin(\frac{\theta}{2})\right]\vec{V}\left[cos(\frac{\theta}{2}) - i\sigma_z sin(\frac{\theta}{2})\right] \tag{17}$$
$$= [V_x\cos\theta + V_y\sin\theta]\sigma_x + [V_y\cos\theta - V_x\sin\theta]\sigma_y + V_z\sigma_z, \tag{18}$$

where a whole bunch of algebra was performed between Eqn. 17 and Eqn. 18. We can now make the identifications:

$$V'_x = V_x \cos\theta + V_y \sin\theta$$
$$V'_y = V_y \cos\theta - V_x \sin\theta.$$

Therefore, if we make the definitions shown in Eqns. 15 and 16, we can perform a three-vector spatial rotation by using $SU(2)$ group members. Sometimes you will hear this type of relation referred to as an isomorphism. For this particular situation the terminology is: $SU(2)$ is isomorphic to $SO(3)$.

You will probably hear that $SU(2)$ "double covers" $SO(3)$. This terminology is due to the fact that the group members of $SU(2)$ are given by $U = e^{i\vec{S}\vec{\theta}} = e^{i\vec{\sigma}\frac{\vec{\theta}}{2}}$. This means that a full rotation in $SU(2)$ requires $\theta = 4\pi$. If we had, instead, defined $U = e^{i\vec{\sigma}\vec{\theta}}$ (i.e. without the factor $\frac{1}{2}$ on $\vec{\sigma}$), the transformation $\vec{V} \to \vec{V}'$ would result in

$$V'_x = V_x cos(2\theta) + V_y sin(2\theta)$$
$$V'_y = V_y cos(2\theta) - V_x sin(2\theta).$$

At this point we would not be able to associate θ as a spatial angle. Instead, we would have to define the spatial angle associated with the $SU(2)$ transformation as $\theta/2$. This is why the factor of $\frac{1}{2}$ was included in the definition of the $SU(2)$ generator. This factor allows the free parameters of $SU(2)$ to be equal to the angles of $SO(3)$.

G. $SU(3)$

I will conclude this section with a very brief look at $SU(3)$. As stated earlier, this group has eight generators. Because of the large number of generators it would be good to have some sort of simple algorithm to find them. One such algorithm relates the generators of $SU(N)$ to those of $SU(N-1)$. I will describe this algorithm for obtaining the $SU(3)$ generators using the $SU(2)$ generators.

Let the generators of $SU(3)$ be denoted as λ_i, where $i = 1, 2, 3, ..., 8$. For each generator of $SU(2)$ (in this case, we are using the Pauli matrices to be the generators), create a matrix that includes that $SU(2)$ generator in the upper left-hand part of the matrix. This gives us λ_1, λ_2, and λ_3.

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The next two matrices we construct will be obtained by taking a 3×3 matrix that has all elements equal to zero, and then placing in the last column and last row, the numbers 1 and 1 symmetrically

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We then construct the next two matrices by placing $-i$ and i symmetrically in the last row and column:

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}.$$

Finally, we put the 2×2 unit matrix in the left top corner of the 3×3 matrix and demand that the matrix is traceless and that the trace of the matrix-squared is equal to 2:

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

These matrices are called the Gell-Mann "λ-matrices" and can be found in most particle physics books. One of the nice features about this simple algorithm is that it allows an easy way to remember the form of the λ-matrices. Note: We could use this same algorithm to construct the generators of $SU(4)$ by starting with our newly created $SU(3)$ generators (if interested, see pages 372-373 of "Quantum Field Theory", by Michio Kaku).

For completeness, the Lie algebra for $SU(3)$ is

$$[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}] = i f_{ijk} \frac{\lambda_k}{2},$$

where the structure constants f_{ijk} are

$$f_{123} = 1,$$
$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2},$$
$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}.$$

It is worth noting that these Gell-Mann λ matrices were constructed in a manner that left, explicitly, $SU(2)$ as a subgroup. The way to see this is by noticing that λ_1, λ_2, and λ_3 are just the Pauli matrices with an extra row and column of zeros thrown in. This means that we can use the λ_1, λ_2, and λ_3 generators to form the $SU(2)$ group. Thus, if we restricted ourselves to transformations that only utilized the λ_1, λ_2, and λ_3 generators, we would only be mixing two out of the three complex vector components.

V. GROUP MULTIPLICATION

A. $SO(2) \times SO(2)$

Let the vectors $\vec{A}$ and $\vec{B}$ be arbitrary two-component real vectors. We know that each these vectors can be transformed such that the scalar length of each vector is invariant under the transformation. In particular, the transformation for real two-component vectors that preserves length is provided by the two-dimensional rotation group. This group, as we have already seen, is called $SO(2)$.

We are interested in constructing something called the "fundamental representation" of $SO(2) \times SO(2)$. This fundamental representation will simply be the direct product of $\vec{A}$ and $\vec{B}$, and will be defined as

$$T^{i,j} = A^i B^j.$$

Since the representation is rather small, it takes little effort to write out the matrix form of T and count the number of independent elements:

$$T^{i,j} = \begin{bmatrix} A^x B^x & A^x B^y \\ A^y B^x & A^y B^y \end{bmatrix} : \text{ 4 independent elements.}$$

We are now interested in finding ways of "breaking up" the fundamental representation into parts that will transform only among themselves. To accomplish this break-up, we will notice that we can create an antisymmetric tensor A out of our fundamental representation T by the simple construction $A^{i,j} = (T^{i,j} - T^{j,i})/2$. We can also make a symmetric tensor $S^{i,j}$ in a similar manner, $S^{i,j} = (T^{i,j} + T^{j,i})/2$. Another quantity that is easy to construct is the trace (dot product): $tr(T^{i,j})$. Since A is antisymmetric it has no trace, but S has a trace, and if we want to break-out the trace from T so that it is independent of the other quantities, we will have to remove this trace from $S^{i,j}$. If we let Q be the symmetric and traceless matrix corresponding to T, then $Q^{i,j} = S^{i,j} - \frac{1}{2}tr(T)$. In summation, we have a fundamental representation T that is broken into three independent parts: An antisymmetric tensor A; a traceless symmetric tensor S, and a trace $tr(T)$. We can write out these tensors and count the number of independent elements:

$$\begin{aligned} A = & \ \frac{1}{2}\begin{bmatrix} 0 & (A^x B^y - A^y B^x) \\ (A^y B^x - A^x B^y) & 0 \end{bmatrix} & : \ 1. \\ Q = & \ \frac{1}{2}\begin{bmatrix} (A^x B^x - A^y B^y) & (A^x B^y + A^y B^x) \\ (A^y B^x + A^x B^y) & (A^y B^y - A^x B^x) \end{bmatrix} & : \ 2. \\ tr(T) = & \ \qquad A^x B^x + A^y B^y & : \ 1. \end{aligned}$$

We cannot decompose this down any further. We say that the fundamental representation has been "reduced" into its "irreducible representations". We count the number of independent elements of each irreducible representation and write

$$SO(2) \otimes SO(2) = 2 \oplus 1 \oplus 1.$$

We say that we have the decomposition of $SO(2) \otimes SO(2)$.

It will be instructive to find the explicit transformation rules for these tensors. Immediately, we can see that the antisymmetric tensor represents the cross product, and that the trace represents the dot product. Since we know that the cross product (in 2-dimension) and dot product are invariant under rotation, then $tr(T)$ and the antisymmetric tensor A are invariants (as is required by them having only one independent element). We only need to worry about how the components of the traceless antisymmetric tensor $Q^{i,j}$ transform:

$$\begin{aligned} (A^x B^x - A^y B^y)' & = \\ C^1(A^y B^y - A^x B^x) & + C^2(A^y B^x + A^x B^y) \\ (A^y B^x + A^x B^y)' & = \\ C^2(A^y B^x - A^x B^y) & + C^1(A^y B^y + A^x B^x) \end{aligned}$$

where $C^1 = (cos^2(\theta) - sin^2(\theta))$, and $C^2 = 2\cos\theta\sin\theta$.

If we now define the vector $\vec{V}_{\text{irreducible}}$ as

$$\vec{V}_{\text{irreducible}} = \begin{pmatrix} V^1 \\ V^2 \\ V^3 \\ V^4 \end{pmatrix} \equiv \begin{pmatrix} A^x B^x - A^y B^y \\ A^y B^x + A^x B^y \\ |\vec{A} \times \vec{B}| \\ \vec{A} \cdot \vec{B} \end{pmatrix}, \quad (19)$$

then the transformation $\vec{V}_{\text{irreducible}} \rightarrow \vec{V}'_{\text{irreducible}}$ can be written

$$\vec{V}'_{\text{irreducible}} = \begin{bmatrix} \begin{bmatrix} C^1 & C^2 \\ C^2 & C^1 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} \vec{V}_{\text{irreducible}}.$$

Thus, we can write the transformation of an irreducible representation as a block diagonal matrix.

Hopefully, you can now see that writing $SO(2) \otimes SO(2) = 2 \oplus 1 \oplus 1$ makes a lot of sense. The term $2 \oplus 1 \oplus 1$ tells us how big the block diagonals are in the irreducible representation.

In short, if we had two real two-component vectors, then we could create the vector $\vec{V}_{\text{irreducible}}$ and any coordinate rotation would result in mixing only the V_1 and V_2 components in Eqn. 19. For two-dimensional real vectors all this group theory is overkill. You have gotten along just fine without all of this mathematical jargon and machinery. The only real reason I wrote all of this stuff for $SO(2) \otimes SO(2)$, is that it is easy and intuitive.

In particle physics we classify the hadrons by there irreducible representations. Later, you will see how all of this machinery relates to the quark composition of the π mesons. First, however, we will see how the decomposition we performed here relates to $SO(N) \otimes SO(N)$.

B. $SO(N) \otimes SO(N)$

To see that the reduction we used for $SO(N) \otimes SO(N)$ (traceless-symmetric, trace, and antisymmetric decomposition) applies generally for $SO(N) \otimes SO(N)$, we just need to show that, in general, $Q^{i,j}$, $tr(T)$, and $A^{i,j}$ transform among themselves.

Since the transformation of symmetric and antisymmetric tensors is written as,

$$(S')^{i,j} = \sum_l^N \sum_m^N O^{i,l} O^{j,m} S^{l,m}$$

$$(A')^{i,j} = \sum_l^N \sum_m^N O^{i,l} O^{j,m} A^{l,m},$$

and $S^{l,m} = S^{m,l}$, and $A^{l,m} = -A^{m,l}$. We can rewrite these as

$$\begin{aligned}(S')^{i,j} &= \sum_l^N \sum_m^N O^{i,l} O^{j,m} S^{m,l} &= (S')^{j,i}\\ (A')^{i,j} &= -\sum_l^N \sum_m^N O^{i,l} O^{j,m} A^{m,l} &= -(A')^{j,i}.\end{aligned}$$

Thus, the symmetric and antisymmetric tensors transform only among themselves. Also, since the trace transforms as

$$\begin{aligned} tr(T') &= \sum_i^N (T')^{i,i} = \sum_{i,j,k}^N O^{j,i} O^{k,i} T^{j,k} \\ &= \sum_{j,k}^N T^{j,k} (O^T O)^{j,k} = \sum_j^N T^{j,j} \\ &= tr(T), \end{aligned}$$

then we have shown that the reduction into symmetric traceless, trace, and antisymmetric tensors form valid representations of $SO(N) \otimes SO(N)$.

C. $SU(2) \otimes \overline{SU(2)}$ flavor symmetry

It is probably more common to see $SU(2) \otimes SU(\bar{2})$ (or even $2 \otimes \bar{2}$) than $SU(2) \otimes \overline{SU(2)}$. The reason I titled this subsection as $SU(2) \otimes \overline{SU(2)}$, rather than the more common notation, is to make it very clear that there is an important distinction being made. In particle physics the over-bar on a particle represents anti-particle. (So, for example, the u quark has as its anti-particle $\bar{u}$.)

The $SU(N)$ transformation properties are different for particles and anti-particles. To see that this should be true, you need to know that the anti-particle scalar-field is obtained by complex conjugation of the particle scalar-field. (For a Dirac field you need, in addition to the conjugation, a factor of the gamma matrix γ^2.) For a particle that respects $SU(N)$ symmetry we have the transformation for the field ϕ^i representing that particle as

$$\phi^i \to \phi'^{\,i} = U^i_j \phi^j,$$

where we have used the convention that $A^i B_i \equiv \sum_i^N A^i B_i$ (i.e. upper and lower indices that match are automatically summed). For a scalar-antiparticle we get

$$\phi^{*i} \to (U^i_j)^* \phi^{*j} = \sum_i^N (U^\dagger)^j_i \phi^{*j},$$

This is a bit of a mess! One way to make this a bit cleaner is to define the anti-particle field for ϕ^i as having a lower index instead and defining the transformation as

$$\phi_i \to \phi'_i = (U^\dagger)^j_i \phi_j.$$

For right now, we just want to know that the $SU(2)$ transformation properties of a particle are different than that of an anti-particle, and that we can keep track of this difference by using an upper index for particles and a lower index for anti-particles.

Lets define two arbitrary two-component complex vectors $\vec{A}$ and $\vec{\bar{B}}$ that have components

$$\vec{A} = \begin{pmatrix} A^u \\ A^d \end{pmatrix}, \ \vec{\bar{B}} = \begin{pmatrix} B_u \\ B_d \end{pmatrix},$$

where we are using the upper index to define a particle and the lower index to define an anti-particle.

The fundamental representation of $SU(2) \otimes SU(2)$ can be written

$$T^i_j = \begin{bmatrix} A^u B_u & A^u B_d \\ A^d B_u & A^d B_d \end{bmatrix} : \text{ 4 independent elements.}$$

In contrast to $SO(N) \otimes SO(N)$, we can no longer form symmetric and antisymmetric tensors that transform only among themselves. We do, however, have the trace that transforms correctly:

$$tr(T') = (T')^i_i = U^i_j (U^\dagger)^k_i T^j_k = (UU^\dagger)^k_j T^j_k = T^j_j = tr(T).$$

This means that we can only form the trace (denoted as ϕ) and traceless ($\phi^i_j = T^i_j - \frac{1}{2} tr(T)$) tensors:

$$\begin{aligned} \phi &= A^u B_u + A^d B_d &: 1 \\ \phi^i_j &= \begin{bmatrix} (A^u B_u - A^d B_d)/2 & A^u B_d \\ A^d B_u & (A^d B_d - A^u B_u)/2 \end{bmatrix} &: 3. \end{aligned}$$

In group theory terminology we say $SU(2) \otimes \overline{SU(2)} = 3 \oplus 1$.

If we make the substitutions $A^u \to u, A^d \to d, B_u \to \bar{u}, B_d \to \bar{d}$, make the meson identifications as given in the PDG, and normalize, then the triplet gives us

$$\begin{aligned} \pi^0 &= (u\bar{u} - d\bar{d})/\sqrt{2} \\ \pi^+ &= u\bar{d} \\ \pi^- &= d\bar{u} \end{aligned}$$

The pions are constructed as elements of the irreducible triplet representation of $SU(2) \otimes SU(\bar{2})$ flavor symmetry. We say that in isospin space, the pions transform among each other. We can make the irreducible representation vector as

$$\vec{V}_{\text{irreducible}} = \begin{pmatrix} \pi^+ \\ \pi^- \\ \pi^0 \\ \text{singlet} \end{pmatrix}$$

What about the singlet? At first glance it looks to be the ω! Well, we don't have the complete story yet. To have a better picture of what is going on, we will need to look at $SU(3) \otimes SU(\bar{3})$.

D. $SU(3)\otimes SU(\bar{3})$

We can construct $SU(3)\otimes SU(\bar{3})$ in a similar manner to that of $SU(2)\otimes SU(\bar{2})$. In fact we can start out by writing $SU(3)\otimes SU(\bar{3}) = \bar{8}\oplus 1$, since we know that there will be nine states and the only thing we can do is form a singlet representation that is the trace, and a representation that is traceless. Immediately we can write

$$\phi = \quad u\bar{u}+d\bar{d}+s\bar{s}$$
$$\alpha^i_j = \begin{bmatrix} \frac{1}{3}(-2u\bar{u}+d\bar{d}+s\bar{s}) & u\bar{d} & u\bar{s} \\ d\bar{u} & \frac{1}{3}(u\bar{u}-2d\bar{d}+s\bar{s}) & d\bar{s} \\ s\bar{u} & s\bar{d} & \frac{1}{3}(u\bar{u}+d\bar{d}-2s\bar{s}) \end{bmatrix}$$

We can identify the singlet as the $\eta' = (u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$, but where did our $\pi^0 = (u\bar{u}+d\bar{d})/\sqrt{2}$ go?

We know that $SU(2)$ flavor symmetry is a "good" symmetry. It certainly is better than $SU(3)$ (since $m_u = 0.002$, $m_d = 0.005$, and $m_s = 0.1$ GeV). It seems reasonable to see if we can form the $SU(2)\otimes SU(\bar{2})$ subgroup explicitly in our octet representation. In fact, we explicitly constructed our Gell-Man λ matrices to maintain $2\otimes\bar{2}$ as a subgroup. It would be somewhat unusual if we could not construct $3\otimes\bar{3}$ with a $2\otimes\bar{2}$ subgroup!

To make $3\otimes\bar{3}$ with an explicit $2\otimes\bar{2}$ subgroup we simply define

$$\phi^i_j = \begin{bmatrix} \frac{1}{3}(u\bar{u}-d\bar{d}) & u\bar{d} & u\bar{s} \\ d\bar{u} & \frac{-2}{3}(u\bar{u}-s\bar{s}) & d\bar{s} \\ s\bar{u} & s\bar{d} & \frac{1}{3}(u\bar{u}+d\bar{d}-2s\bar{s}) \end{bmatrix}$$

Once we make the meson identifications as given in the PDG, and normalize, then the octet gives us

$$\begin{aligned} \eta &= (u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6} \\ \pi^0 &= (u\bar{u}-d\bar{d})/\sqrt{2} \\ \pi^+ &= u\bar{d} \\ \pi^- &= d\bar{u} \\ K^+ &= u\bar{s} \\ K^- &= s\bar{u} \\ K^0 &= d\bar{s} \\ \bar{K}^0 &= s\bar{d}. \end{aligned}$$

Note: Since $\frac{2}{3}(u\bar{u}–s\bar{s}) = \pi^0+\eta$, then the $\frac{-2}{3}(u\bar{u}–s\bar{s})$ term is not independent, and does not belong in our particle listing.

E. The $SU(3)\otimes SU(3)\otimes SU(3)$ baryons

So far we have only concerned ourselves with the $q\bar{q}$ mesons. If we take three quarks we can construct baryons.

Just as in the case of $SO(3)\otimes SO(3)$ we can form a vector from a rank 2 tensor. In the case of $SU(3)\otimes SU(3)$ the resulting vector will have the transformation characteristics of $SU(3)$. In particular, we can form the vectors

$$\phi_k = \phi^{i,j}\epsilon_{ijk}$$

and

$$\phi^k = \phi_{i,j}\epsilon^{ijk}.$$

These vectors within $SU(3)$ are the complex analog of $\vec{A}\times\vec{B}$ within $SO(3)$. Note, this only gives a non-zero result when $\phi^{i,j}$ or $\phi_{i,j}$ is antisymmetric in $i\leftrightarrow j$.

One thing that is important for you to keep in mind: The vector $\phi_k = \phi^{i,j}\epsilon_{ijk}$, transforms like an antiparticle transforms. We are NOT saying that the vector ϕ_k represents an antiparticle. We are just saying that ϕ_k has the same transformation properties under $SU(3)$ as an antiparticle would have.

We can always form objects that have definite symmetry properties under interchange of the upper indices, and definite symmetry properties under interchange of the lower indices. Pay careful attention, this is the important part: for any two upper (lower) indices that are antisymmetric under interchange, we can reduce the number of indices by one and lower (raise) that index by the operation $\phi_k = \phi^{i,j}\epsilon_{ijk}$ ($\phi^k = \phi_{i,j}\epsilon^{ijk}$). This means that we only have to consider irreducible representations that are totally symmetric in lower and upper indices!

Let me form the direct product representation of $SU(3)\otimes SU(3)\otimes SU(3)$ as

$$T^{i,k,j} = \phi^i\phi^j\phi^k$$

We can obtain an irreducible representations that are

- totally symmetric: $\phi^{i,j,k}$
- $\phi = \phi^k_k = \phi^{i,j,k}\epsilon_{ijk}$ (antisymmetric trace)
- antisymmetric in $i\leftrightarrow j$: $\phi^k_m = \phi^{i,j,k}\epsilon_{ijm}$ traceless
- antisymmetric in $i\leftrightarrow k$: $\phi^j_n = \phi^{i,j,k}\epsilon_{ink}$ traceless

But what about antisymmetric in $j\leftrightarrow k$? It turns out that the states that are antisymmetric in $j\leftrightarrow k$ can be made out of the other states. (See handout.)

In short we can write

$$\phi^i\otimes\phi^j\otimes\phi^k = \phi^{i,j,k}\oplus\phi^k_m\oplus\phi^j_n\oplus\phi,$$

or

$$3\otimes 3\otimes 3 = 10_S\oplus 8_M\oplus 8_M\oplus 1_A,$$

where we have a symmetric decuplet, two mixed symmetry octets, and an antisymmetric singlet. We can even write:

$$(1,0)\otimes(1,0)\otimes(1,0) = (3,0)\oplus(1,1)\oplus(1,1)\oplus(0,0),$$

where the notation (m,n) means m upper indices and n lower indices.

Note: We already found that $\phi^k_m\to 8$. To determine $\phi^{i,j,k}\to 10$ we simply count the possible quark states:

- no indices matching → 1 state: $|uds>$
- two indices matching → 6 states: $|uud>$, $|uus>$, $|ddu>$, $|dds>$, $|ssu>$, $|ssd>$
- all indices matching → 3 states: $|uuu>$, $|ddd>$, $|sss>$.

F. A couple more $SU(3)$ decompositions

Let's do a couple more decompositions.

If I want to do a group multiplication of $3 \otimes 3$ I get

$$\phi^i \otimes \phi^j = \phi^{i,j} \oplus \phi_k.$$

Since $\phi^{i,j}$ is symmetric under $i \leftrightarrow j$, there are six independent states. Thus,

$$3 \otimes 3 = 6 \oplus \bar{3}.$$

Note: we obtain a $\bar{3}$ decomposition instead of 3 since this triplet transforms the same way as an antiparticle triplet.

If I want to do a group multiplication of the baryon octet with the meson octet, I get

$$\phi^i_j \otimes \phi^k_l = \phi^{i,k}_{j,l} \oplus \phi^{i,k,m} \oplus \phi_{j,l,n} \oplus \phi^m_o \oplus \phi^p_n \oplus \phi.$$

To count the number of states within $\phi^{i,k}_{j,l}$ we notice that there are six ways to have (i,k) symmetric and six ways to have (j,l) symmetric. This implies $6 \cdot 6 = 36$ states symmetric in $i \leftrightarrow k$ and $j \leftrightarrow l$. We just have to subtract out all of the traces to find the number of states within $\phi^{i,k}_{j,l}$.

To count the number of traces we notice that if one upper and one lower index are summed to form a trace, there are three ways to choose the remaining upper index and three ways to choose the remaining lower index. This means that there are $3 \cdot 3 = 9$ states that include at least one trace. Now, to find the number of states within $\phi^{i,k}_{j,l}$ we just need to subtract the nine traces from the 36 states symmetric in $i \leftrightarrow k$ and $j \leftrightarrow l$. Thus $\phi^{i,k}_{j,l}$ contains 27 states.

We can now make the identification:

$$8 \otimes 8 = 27 \oplus 10 \oplus \bar{10} \oplus 8 \oplus 8 \oplus 1.$$

So, finding multiplets within $SU(3)$ is a fairly straightforward process. The most challenging part is counting the number of states within any given irreducible representation.

VI. CONCLUSION

In this document, an introduction to continuous groups has been presented. The symmetry groups have many applications in particle physics, particularly $SU(2)$ flavor symmetry (isospin), $SU(3)$ flavor symmetry (the quark model with u, d and s quarks), and $SU(3)$ color symmetry (with R, G, and B color charges).

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