Making sense of field-theory Lagrangians for η and η' photoproduction

M. Dugger
Arizona State University, Tempe, AZ 85287-1504

Interaction Lagrangians for η and η' photoproduction are discussed.

I. INTRODUCTION

The purpose of this document is to help the student of nuclear/particle physics obtain an intuitive feel for the terms seen in Lagrange densities for η and η' photoproduction. For this reason, this document is designed to only show the most relevant aspects of field theory. Along the way I will show how to obtain a non-relativistic potential from a relativistic Lagrange density. Once the connection from relativistic Lagrange densities to non-relativistic potentials is made, we will see how the nucleon-nucleon potential can be related to a single pion exchange model. We will then piece together interaction terms commonly found in Lagrange densities for η and η' photoproduction.

II. REVIEW OF LAGRANGIANS

A. The classical discrete Lagrangian

It is assumed that the concept of generalized coordinates are understood. Also, in all that follows, generalized forces are assumed to be are derivable from potentials. This allows us to write the Lagrangian as

$$L = T - V,$$

where T is the kinetic energy and V is the potential energy. It is important to note that the Lagrangian is a function of the generalized coordinate q, and the time derivative of the generalized coordinate, $\dot{q}$. To make this statement explicit the Lagrangian is often written as $L(q, \dot{q})$.

The equations of motion are derived using the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0.$$

The Euler-Lagrange equation may be derived by taking the extremum $\delta S = 0$, where $S = \int L dt$. The generalized momentum p_k is given by

$$p_k \equiv \frac{\partial L}{\partial \dot{q}}. \tag{1}$$

The Euler-Lagrange equation now implies that any coordinate q_k that does not occur explicitly within L has a corresponding momentum p_k that is a constant of the motion.

For the very simple case where $T = \frac{1}{2}m\dot{q}^2$, and $V = mgq$, we get $L = \frac{1}{2}m\dot{q}^2 - mgq$, and the equation of motion becomes

$$m\ddot{q} = mg.$$

While the generalized momentum $p_q = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$. We have just found that the acceleration is equal to g, $p_q = m\dot{q}$, and that all momenta not associated with the generalized coordinate q are constant. Told you this would be a very simple example!

B. The relativistic field Lagrangian

For a field, the Lagrangian is defined over the region of space occupied by the field. For this reason, it is more convenient to work with the "Lagrange density" $\mathcal{L}$, than with L directly. This Lagrange density is related to the Lagrangian in the following manner

$$L = \int_{-\infty}^{+\infty} d^3x \mathcal{L}.$$

In this case, the Euler-Lagrange equation is modified to read

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0,$$

where $\partial^\mu \equiv \partial/\partial x^\mu$, $\phi \equiv \phi(x)$, and as before, is constructed to minimize the action $S = \int L dt = \int \mathcal{L} d^4x$. Also, in all that follows, any upper index that matches a lower index is to be summed (i.e. $A^\mu B_\mu \equiv \sum_\mu A^\mu B_\mu$). It is important to note that the Lagrangian has units of energy and must be a scalar.

After looking at the Euler-Lagrange equation, it might be tempting to write $\mathcal{L}(\phi, \partial^\mu \phi)$, but that would be wrong. For a field, the Lagrange density $\mathcal{L}$ is a function of the field amplitude ϕ and the time-derivative of that field $\dot{\phi}$. This is often written explicitly as $\mathcal{L}(\phi, \dot{\phi})$.

The "momentum density" $\pi(x)$ of the field is given by

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)},$$

a direct analogy to the classical case shown in Eqn. 1.

It is important to note that the generalized coordinate is an amplitude ϕ evaluated at some point in space x, and is written $\phi(x)$. If we were to analyze a string under tension as a one dimensional field, we could write the amplitude as a physical displacement. However, if we analyze the electromagnetic four-vector potential $A^\mu(\vec{x})$ as a field, we would have the amplitude of A^μ at some position $\vec{x}$ as the generalized field coordinates. In the case of the $A^\mu(\vec{x})$ field, there is no physical coordinate displacement associated with the generalized coordinate. We could, however, if so desired, find the electric and magnetic fields associated with $A^\mu(\vec{x})$ through the relations:

$$\begin{aligned}\vec{E} &= -\vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t},\\ \vec{B} &= \vec{\nabla}\times\vec{A}.\end{aligned}$$

For our purposes, we wish to look at the scalar field $\phi(x)$ that has the free Lagrangian density $\mathcal{L}_0$

$$\mathcal{L}_0 = \frac{1}{2}[(\partial^\mu\phi)^2 - m^2\phi^2],$$

where for any four-vector quantity $(Z^\mu)^2 \equiv Z^\mu Z_\mu$. After solving the Euler-Lagrange equation, we find the equation of motion

$$(\partial^\mu\partial_\mu + m^2)\phi = 0.$$

This is simply the Klein-Gordon equation. This means that our free Lagrangian density $\mathcal{L}_0$ describes a relativistic spin-zero field. (Note: Throughout this document, the units are taken such that $\hbar = c = 1$.)

Sometimes, the determination of the equation of motion is described as first quantization. However, since we have not quantized anything yet, this is somewhat a misnomer. To quantize the field, we need to impose a quantization scheme. The confusion regarding first quantization comes from the fact that the Klein-Gordon equation applied to a particle is, in fact, quantum mechanics. Yet, if the Klein-Gordon equation is applied to a field, we are still doing classical relativistic physics. Anyway, I only mention this because some people like the terms "first-" and "second-quantization", and other do not.

C. Quantizing the field

The position-momentum commutators for the quantum mechanics of point particles are as follows:

$$\begin{aligned}[x, p_x] &= i\\ [x, y] &= [p_x, p_y] = 0.\end{aligned}$$

For quantum fields it seems reasonable that similar relations should hold true. However, since the field is located over a region, we expect that the commutation relations for fields will take this into account.

One canonical quantization scheme is to force the commutation relations between the generalized coordinate and the generalized momentum-density to be

$$\begin{aligned}[\phi(\vec{x},t_0), \pi(\vec{y},t_0)] &= i\delta^{(3)}(\vec{x}-\vec{y}),\\ [\phi(\vec{x},t_0), \phi(\vec{y},t_0)] &= [\pi(\vec{x},t_0), \pi(\vec{y},t_0)] = 0,\end{aligned}$$

where it is important to note that each of these field operators are taken to be at the same time t_0. These are commonly referred to as "equal time commutation relations". If the time of the fields are not identical, then these commutation relations do not hold.

After doing a bunch of math we would end up with (for the Klein-Gordon field):

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}}\left(a_{\vec{p}}\, e^{-ip^\mu x_\mu} + a^\dagger_{\vec{p}}\, e^{ip^\mu x_\mu}\right),$$

where

$$[a_{\vec{p}}, a^\dagger_{\vec{p}\,'}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}\,').$$

We have fields $\phi(\vec{x})$ with creation and annihilation operators. This means that these fields do NOT represent quantum-mechanical states. These fields, instead, are themselves operators. This is one of the key differences between regular quantum mechanics of point particles, and quantum field theory. When you solve the Klein-Gordon equation for a point particle, you end up with a state vector. However, when you solve (and quantize) the Klein-Gordon equation in field theory, you end up with an operator.

What does this field operator do? The field operator $\phi(\vec{x})$ will create or annihilate a particle at position $\vec{x}$. That is pretty much it.

In particular, if a state of momentum $\vec{p}$ is defined so that

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}}|0\rangle,$$

then

$$\begin{aligned}\langle\vec{p}| &= \langle 0|a_{\vec{p}}\sqrt{2E_{\vec{p}}},\\ a_{\vec{p}\,'}|\vec{p}\rangle &= (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}\,')\sqrt{2E_{\vec{p}}}\,|0\rangle,\\ \langle\vec{p}|a^\dagger_{\vec{p}\,'} &= \langle 0|(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}\,')\sqrt{2E_{\vec{p}}},\\ a_{\vec{p}}|0\rangle &= \langle 0|a^\dagger_{\vec{p}} = 0\end{aligned}$$

and

$$\phi(\vec{x})|0\rangle = \int\frac{d^3p}{(2\pi)^3 2E}e^{-ip^\mu x_\mu}|\vec{p}\rangle$$

and

$$\langle 0|\phi(\vec{x})|\vec{p}\rangle = e^{ip^\mu x_\mu}.$$

Don't worry too much about the factors of $\sqrt{2E}$. The important thing is to notice that the field operators create and annihilate states.

III. THE PROPAGATOR

So that we can make sense of what propagators are and how they are related to the Lagrange density $\mathcal{L}$, I will show how the propagator is found using two different

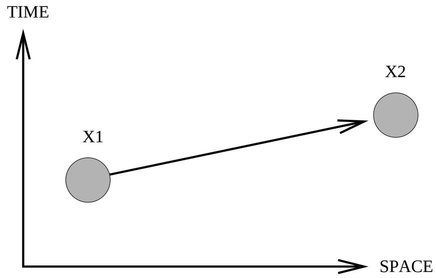

FIG. 1: Feynman diagram for a propagator from point x_1 to point x_2.

methods. The first method will clearly show that the propagator describes a state being created at some point in space-time and then annihilated at some other space-time point. The second method will clearly show how the propagator is related to the free Lagrange density $\mathcal{L}_0$.

A. Propagator part 1

In this method we will define the propagator $D(x_1-x_2)$ as

$$D(x_1 - x_2) = \langle 0|\phi(\vec{x_2}, t)\phi(\vec{x_1}, 0)|0\rangle,$$

where, in this case, I have taken $t > 0$.

The propagator, as defined above, says that a state is created as position x_1 at time $t = 0$, and is annihilated at position x_2 at time t. Working out the equation, we find that the only term that can contribute contains $\langle 0|a_{\vec{p}}\, a^\dagger_{\vec{q}}|0\rangle = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{q})$ and leads to

$$D(x_1 - x_2) = \int \frac{d^3p}{(2\pi)^3 2E} e^{-ip\cdot(x_1-x_2)}, \tag{2}$$

where the dot product of two four-vectors A^μ, B_μ is defined such that $A \cdot B = A^\mu B_\mu$.

We are now ready to see our very first Feynman diagram in Fig. 1! We can see that the figure makes a lot of sense. The figure says that a field propagates from point x_1 to point x_2. We can even associate an equation with this propagation.

B. Propagator part 2

Another way to define the propagator for our scalar field theory is by taking the equation of motion and "inverting it". The equation of motion for our scalar field theory is the Klein-Gordon equation:

$$(\partial^\mu\partial_\mu + m^2)\phi = 0.$$

If I were to write this as $A\phi = 0$, I could define A^{-1} as such:

$$AA^{-1} = \delta^{(4)}(x_1 - x_2).$$

That is what I mean by inverting the equation of motion. For the field of interest we define the propagator $D(x_1 - x_2)$ through the relation

$$-i(\partial^2 + m^2)D(x_1 - x_2) = \delta^{(4)}(x_1 - x_2).$$

To find $D(x_1 - x_2)$ it is helpful to remember that the δ-function can be written

$$\delta^{(4)}(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x_1-x_2)}, \tag{3}$$

and that

$$(\partial^2 + m^2)e^{ik\cdot(x_1-x_2)} = (-k^\mu k_\mu + m^2)e^{ik\cdot(x_1-x_2)}.$$

By inspection we can now write the propagator as

$$D(x_1 - x_2) = i\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x_1-x_2)}}{k^2 - m^2 + i\epsilon}, \tag{4}$$

where ϵ represent an infinitesimal shift in the imaginary part of the denominator and is a technical detail we will not worry about. But wait! This does not look like our previous result. How is this propagator in Eqn. 4 the same as the one in Eqn. 2?

To see that Eqn. 4 might represent Eqn. 2, we should notice that Eqn. 2 does not have the zero component of k integrated over. Perhaps, if we integrate Eqn. 4 over the variable k^0 we will obtain Eqn. 2. To do this integration requires that we use the method of contours of complex analysis. We don't want to get bogged down in too much math, so I'll just state the result for the case when $t_2 > t_1$

$$D(x_1 - x_2) = \int \frac{d^3p}{(2\pi)^3 2E} e^{-ip\cdot(x_1-x_2)},$$

which is precisely Eqn. 2.

Another issue that might give you pause, is that the integrand of the propagator given in Eqn. 4 is infinite when $k^2 = m^2$! Isn't the invariant quantity for a four-momentum the mass? That is true, however, this four-momentum is said to be off the "mass shell". In classical quantum mechanics of point particles, the energy is not always conserved. For quantum field theory, the 4-momentum is conserved (working with plane waves $\Rightarrow \Delta p^\mu = 0$ and $\Delta x \to \infty$), but the mass of the particles will vary from that of "free particles". This variation of the mass is called being off the mass shell.

In the field theory that we are interested in, we are more concerned with the form of the propagator given in Eqn. 4. Note: it is possible for the propagator to be defined such that the factor of i is removed from Eqn. 4. I have elected to keep the factor of i so that our two methods of determining the propagator are consistent. Also, it is useful to note that the Fourier transform of Eqn. 4 to momentum-space can be written as

$$\tilde{D}(k) = \frac{i}{k^2 - m^2 + i\epsilon}.$$

I want you to notice that we only needed the free Lagrange density $\mathcal{L}_0$ to determine the propagator. When you see a Lagrange density $\mathcal{L}$, you should mentally separate the free part $\mathcal{L}_0$ from the interaction part $\mathcal{L}_{int}$, so that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, and notice that the free part $\mathcal{L}_0$ determines the field propagators.

Aside: I want to make something clear regarding terminology. Sometimes you will here me refer to the free Lagrange density $\mathcal{L}_0$ as being the kinetic part of the Lagrangian. This is a misnomer. The kinetic part of $\mathcal{L}$ for our simple theory is $(\partial\phi)^2/2$ and not $[(\partial\phi)^2 - m^2\phi^2]/2$. The mass term is more accurately associated with potential energy. So, please excuse me if I say that $\mathcal{L}_0$ is the kinetic part of the Lagrange density. Some habits are hard to break.

Now that we have associated the propagator with the free part of $\mathcal{L}$, it is time to ask about the interaction part of the Lagrange density.

IV. THE VERTEX

I will describe a first order process that has as its Lagrange density

$$\begin{aligned}\mathcal{L} = \ & \tfrac{1}{2}[(\partial\phi_a(\vec{x}))^2 - m_a^2\phi_a^2(\vec{x})] + \\ & \tfrac{1}{2}[(\partial\phi_b(\vec{x}))^2 - m_b^2\phi_b^2(\vec{x})] + \\ & \tfrac{1}{2}[(\partial\phi_c(\vec{x}))^2 - m_c^2\phi_c^2(\vec{x})] - \ g\phi_a(\vec{x})\phi_b(\vec{x})\phi_c(\vec{x}).\end{aligned}$$

You might be saying to yourself: This equation is too long! If I know that the fields are scalar (Klein-Gordon) then the $\mathcal{L}_0$ part is not even needed. We could get by just as well by specifying the interaction term and stating that the fields $\phi_a\phi_b\phi_c$ are scalar. You would be right in making such a statement. This is why, in many cases, the free part is not even written. It is easier just to write

$$\mathcal{L}_{int} = -g\phi_a\phi_b\phi_c.$$

You will also often encounter the case where the subscript *int* is removed, and see things like

$$\mathcal{L} = -g\phi_a\phi_b\phi_c,$$

to describe a Lagrange density. So, if you see a theorist write down a field that only contains interaction parts, it is assumed that the free part has been conveniently suppressed to save on writing a bunch of free field Lagrangians.

To get an intuitive idea of what an interaction term in the Lagrange density tells us, consider three different fields $\phi_a(\vec{x})$, $\phi_b(\vec{x})$, and $\phi_c(\vec{x})$, representing three different particles (denoted as A, B, and C) with the mass of A greater than the mass of $B + C$. Each of these fields have their own $\mathcal{L}_0$, but have a common interaction term $\mathcal{L}_{int} = -g\phi_a(\vec{x})\phi_b(\vec{x})\phi_c(\vec{x})$. Lets just naively find

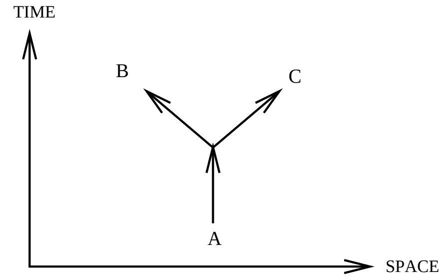

FIG. 2: Vertex at point x.

$\langle\vec{p_b}, \vec{p_c}|\ L_{int}|\vec{p_a}\rangle$:

$$\langle\vec{p_b}, \vec{p_c}|\ L_{int}|\vec{p_a}\rangle = \langle\vec{p_b}, \vec{p_c}| \int d^4x\mathcal{L}_{int}|\vec{p_a}\rangle =$$
$$-g\langle\vec{p_b}, \vec{p_c}| \int d^4x\phi_b(\vec{x})\phi_c(\vec{x})\phi_a(\vec{x})\ |\vec{p_a}\rangle$$

When we multiply the fields together the only term that will survive will have a factor of $b^\dagger_{\vec{q}}\ c^\dagger_{\vec{r}}\ a_{\vec{s}}$. This will give us, after a bit of math,

$$\begin{aligned} & -g\ \ \langle\vec{p_b}, \vec{p_c}| \int d^4x\phi_b(\vec{x})\phi_c(\vec{x})\phi_a(\vec{x})\ |\vec{p_a}\rangle \\ = & \ -g\ \ \delta^{(4)}(p_a - (p_b + p_c)), \end{aligned} \tag{5}$$

where the δ-function tells us to conserve momentum and energy at the vertex.

I will now try and make the same sort of statement in a different way. The method I am about to describe will involve more math but better describes how to calculate something called the $\mathcal{M}$-matrix amplitude that we associate with Feynman diagrams.

We will now draw our next Feynman diagram. This Feynman diagram can be found in Fig. 2 and says that particle A decays into particles B and C at a vertex.

I will define an amplitude for a processes (e.g. $A \to B+C$) as Z. To determine Z we associate a factor of $-ig$ with the vertex, propagate all of the lines, and integrate over the vertex position (remember we are dealing with plane waves $\Rightarrow$ vertex can be anywhere):

$$Z = (-ig)\int d^4xD(a-x)D(x-b)D(x-c) =$$
$$-ig\iiiint\frac{d^4k_ad^4k_bd^4k_ci^3e^{-ix\cdot(k_a-k_b-k_c)}e^{i(k_a\cdot a-k_b\cdot b-k_c\cdot c)}}{(2\pi)^{12}(k_a^2-m_a^2)(k_b^2-m_b^2)(k_c^2-m_c^2)}d^4x,$$

where I have collected terms in the exponent common in x. When we integrate over the x variable and notice that this is just a definition of the δ-function (see Eqn. 3), we obtain

$$Z = -ig(2\pi)^4\delta^{(4)}(k_a - k_b - k_c)D(a)D(b)D(c).$$

To obtain $-i\mathcal{M}$ from Z, we just throw away the overall momentum-energy conserving δ-function (along with the

$(2\pi)^4$) and all propagators associated with external lines. For our simple example we get

$$-iM = -ig.$$

We have just seen that it is natural to associate the coupling constant and momentum-energy conserving δ-function at an interaction vertex, with the interaction part of a Lagrange density $\mathcal{L}_{int}$ given as the product of fields, $\mathcal{L}_{int} = -g\phi_a(\vec{x})\phi_b(\vec{x})\phi_c(\vec{x})$. Moreover, we have seen how the diagram is associated with something called the $\mathcal{M}$-matrix, and that the coupling constants of the fields gives a measure of the reaction strength within $\mathcal{M}$.

Now when you see a Lagrange density, you should think of the interaction terms as describing several fields being created and annihilated at some point in space-time.

A. Identical particles at the vertex

In the previous subsection we had three different fields (ϕ_a, ϕ_b, ϕ_c) that interacted at a vertex. If, instead, we had one field with the Lagrange density

$$\mathcal{L} = \frac{1}{2}[(\partial\phi)^2 - m^2\phi^2] + g\phi^3,$$

then, when we form

$$\langle\vec{p_1},\vec{p_2}|\, L_{int}|\vec{p_3}\rangle = \langle\vec{p_1},\vec{p_2}| \int d^4x\mathcal{L}_{int}|\vec{p_c}\rangle = \\ -g\langle\vec{p_1},\vec{p_2}| \int d^4x\phi(\vec{x})\phi(\vec{x})\phi(\vec{x})\,|\vec{p_3}\rangle,$$

we would not be able to distinguish which field creates or annihilates any of the particular p_1, p_2, p_3 states. We would have three ways to choose the first field, two ways to choose the second field, and one way to choose the last field. For this reason, we obtain a factor of 3! when compared to Eqn. 5:

$$\begin{aligned} -g\ \ &\langle\vec{p_1},\vec{p_2}| \int d^4x\phi(\vec{x})\phi(\vec{x})\phi(\vec{x})\,|\vec{p_3}\rangle \\ = \ -g\ \ &3!\,\delta^{(4)}(p_3 - (p_1 + p_2)). \end{aligned}$$

Many times you will see $\mathcal{L}_{int} = -g\phi^3/(3!)$, where the factor $1/(3!)$ is put in so that we don't need to worry about pesky factors of 3! in the Feynman rules.

A more complete description of how Z is obtained in terms of the propagators for the $\mathcal{L}_{int} = -g\phi^3/(3!)$ theory can be found in the appendix.

V. SECOND ORDER INTERACTION

I will describe a second order process (each vertex represents an order, see appendix) describing the elastic scattering of two scalar particles $A + B$ with the interaction Lagrange density given as $\mathcal{L}_{int} = -g\phi_a\phi_b\phi_c$. The

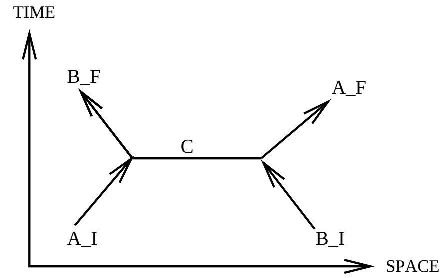

FIG. 3: Feynman diagram for $A + B \rightarrow A + B$. The initial and final positions are labeled: A_I (initial position for particle A); B_I (initial position for particle B); A_F (final position for particle A), and B_F (final position for particle B).

process we will investigate is typically written $A + B \rightarrow A + B$.

The second-order Feynman diagram for $A+B \rightarrow A+B$ is shown in Fig. 3, where I have labeled the various fields in terms of initial and final states.

As was done previously, I will associate a factor of $-ig$ with each vertex, propagate all of the lines, and integrate over the vertex positions to obtain:

$$Z = (-ig)^2\int d^4x_1\int d^4x_2 D_{a_Ix_1}D_{x_1b_F}D_{x_1x_2}D_{b_Ix_2}D_{x_2a_F}, \tag{6}$$

where I have defined $D(x - y) \equiv D_{xy}$ to save space. Integrating over x_1 we get

$$Z = (-ig)^2\int d^4x_2 D(a_I)D(b_F)D(x_2)D_{b_Ix_2}D_{x_2a_F} \times \\ (2\pi)^4\delta^{(4)}(p_{a_I} - p_{b_F} - p_c).$$

Now integrating over x_2 to obtain

$$Z = (-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{iD(a_I)D(b_F)D(b_I)D(a_F)}{k_c^2 - m_c^2} \times \\ (2\pi)^4\delta^{(4)}(p_{a_I} - p_{b_F} - p_c)(2\pi)^4\delta^{(4)}(p_{b_I} - p_{a_F} - p_c).$$

After performing the last integral:

$$Z = -ig^2\frac{D(a_I)D(b_I)D(b_F)D(a_F)}{(p_{a_I} - p_{b_F})^2 - m_c^2} \times \\ (2\pi)^4\delta^{(4)}(p_{a_I} + p_{b_I} - p_{a_F} - p_{b_F}).$$

Now, throw away the external field propagators and the factor $(2\pi)^4\delta$-function and set to $-i\mathcal{M}$ to get

$$\mathcal{M} = \frac{g^2}{(p_{a_I} - p_{b_F})^2 - m_c^2}.$$

From our examples of how to calculate Feynman diagrams, it appears that there are common physical situations that might give the same general results. This is, in fact, the case. We don't need to calculate all of the integrals. Instead, we can create a list of rules that are called "Feynman rules" for dealing with Feynman diagrams and make our lives a little bit easier. Please see the handout that goes with this document. It shows the Feynman rules for the theory presented here, along with the $\mathcal{M}$-matrix calculations I have shown, along with other Feynman diagram calculations I have not shown.

Disclaimer: The method I used in this document to find the $\mathcal{M}$-matrix are only guaranteed to work for the specific calculations shown. To obtain $\mathcal{M}$ for Feynman diagrams different from what I have shown, you will have to use established Feynman rules.

Now that we have the $\mathcal{M}$-matrix for two-body scattering we need to see how this is related to the physical measurements of lifetime and cross-section. The connection will be shown in the next section. After which, we will be concerned with how a potential energy term like $V \propto e^{-mr}/r$ comes from the interaction Lagrange density.

VI. THE $\mathcal{M}$-MATRIX CONNECTION TO THE PHYSICAL OBSERVABLES OF LIFETIME AND CROSS-SECTION

In what follows, I will show the relation between $\mathcal{M}$ and physical observables for some simple reactions. I will state these relations without proof.

First we need to define a few quantities:

- $\tau \equiv$ the mean lifetime of a particle,
- $\Gamma \equiv$ the decay rate of a particle,
- $\sigma \equiv$ the cross section for a reaction,
- $d\sigma/d\Omega \equiv$ the differential cross section.

A. Decay rate

The relationship between the decay rate and lifetime is given simply as

$$\Gamma = \frac{1}{\tau},$$

and is related to the number of particles N through the relation $N(t) = N(0)e^{-\Gamma t}$.

We now want to relate the decay rate to the $\mathcal{M}$-matrix for a two-body decay $1 \rightarrow 2 + 3$ in the center-of-mass frame:

$$\Gamma = \frac{S|\vec{p_f}|}{8\pi m_1^2}|\mathcal{M}|^2,$$

where $\vec{p_f}$ is the momentum of either final particle and S represents a symmetry factor that we don't have to worry about right now. For the reaction $A \rightarrow B + C$ we have $\mathcal{M} = g$ (and the symmetry factor $S = 1$ in this case), so

$$\Gamma = \frac{|\vec{p_b}|}{8\pi m_a^2} g^2,$$

where $\vec{p_b}$ $(-\vec{p_c})$ is determined by energy-momentum conservation.

B. Differential cross section

The center-of-mass differential cross section for the reaction $1 + 2 \rightarrow 3 + 4$ is given by

$$\frac{d\sigma}{d\Omega} = \left[\frac{S}{(8\pi)^2}\right]\left[\frac{|\mathcal{M}|^2}{(E_1 + E_2)^2}\right]\frac{|\vec{p_f}|}{|\vec{p_i}|},$$

where $\vec{p_i}$ $(\vec{p_f})$ is the momentum of either incoming (outgoing) particle. In our example of $A + B \rightarrow A + B$ we have

$$\mathcal{M} \;=\; \left[\frac{g^2}{t^\mu t_\mu - m_c^2}\right],$$

where $t^\mu = p^\mu_{a_I} - p^\mu_{b_F}$, and represents the momentum transfer to particle C in Fig. 3. If I set $m_a = m_b$ we get

$$\mathcal{M} \;=\; \left[\frac{-g^2}{\vec{t} \cdot \vec{t} + m_c^2}\right].$$

Thus

$$\frac{d\sigma}{d\Omega} \;=\; \left[\frac{-g^2}{\vec{t} \cdot \vec{t} + m_c^2}\right]^2 \left[\frac{1}{(8\pi)^2 (E_1 + E_2)^2}\right],$$

(Note: In the the center-of-mass frame $|\vec{p_i}| = |\vec{p_f}|$ for the reaction $A + B \rightarrow A + B$.)

VII. POTENTIAL AND $\mathcal{L}$

To determine the form of the potential using our $\mathcal{M}$-matrix, we will compare our cross section in the non-relativistic limit to the cross section we would obtain using the non-relativistic Born approximation.

For a reaction of the type $A + B \rightarrow A + B$, we can write the non-relativistic Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{m_r^2}{(2\pi)^2}|\tilde{V}(q)|^2, \tag{7}$$

where m_r is the reduced mass $(m_r = m_a m_b/(m_a + m_b))$ and $\tilde{V}(q)$ is the potential in momentum space. Therefore, in the non-relativistic limit, we can make the identification

$$\tilde{V}(q) \rightarrow \frac{1}{4 m_a m_b}\mathcal{M}. \tag{8}$$

Thus, we just need to perform a Fourier transform to obtain $V(\vec{r})$ from $\mathcal{M}$. For our $A + B \rightarrow A + B$ reaction we obtain (remembering that we have set $m_a = m_b$)

$$\begin{aligned} V(r) \;&=\; -\frac{g^2}{4m_a^2}\int \frac{d^3 q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^{\,2} + m^2} \\ &=\; -\left(\frac{g^2}{16\pi m_a^2}\right)\frac{e^{-m_c r}}{r}. \end{aligned} \tag{9}$$

The first thing to notice is that the potential is attractive (minus sign in front of the right hand side of Eqn. 9). In the limit where $m_c \to 0$ we would get the potential $V(r) \propto 1/r$. That is, the potential for the case where $m_c \to 0$, looks like what we are accustomed to for gravity and electromagnetism. For the case where $m_c \neq 0$, the inverse radial dependence is "damped" by the exponential factor $e^{-m_c r}$.

You should notice that, for our simple theory, the momentum dependence of $\mathcal{M}$ was determined by the propagator of the C particle, while the coupling g was determined by the vertex. Thus, both the free part (that determines the propagator) and the interaction part (that determines the vertex couplings) are important in relating the potential to the Lagrange density $\mathcal{L}$.

You should not be too surprised that the potential for the interaction of a scalar field with a fermion field ($\mathcal{L}_{int} = g\,\overline{\psi}\,\phi\,\psi$, where the over-bar on ψ is a technical detail we won't worry about right now) yields $V(r) \propto e^{-m_c r}/r$. After all, the propagator for the ϕ-field is the same as we have seen in our ABC theory. This becomes more interesting if we were to make a strong-force model where baryons (three-quark composite particles like protons and neutrons) interact with scalar mesons. In fact, such theories have been proposed, where one particularly famous model came from Hideki Yukawa. The potential $V(r) \propto e^{-m_c r}/r$ is commonly called the "Yukawa potential" and represents a major achievement in the early theoretical understanding of the nucleon-nucleon force.

Unfortunately, the nucleon-nucleon force is not quite so nice. For one thing, the most common mesons are not scalar particles but, instead, pseudo-scalar. This means that the parity of these particles change sign under a "space-inversion" ($\vec{r} \to -\vec{r}$).

A. One-pion exchange potential for proton-proton scattering

In relativistic quantum mechanics, anti-particles are possible, and this creates a new degree of freedom that is not seen in the non-relativistic theory. In non-relativistic quantum mechanics, a fermion can be represented by a two component spinor, whereas in a relativistic treatment, a four-component spinor is found that describes both the spin and particle/anti-particle nature of the field excitation. As a reminder of relativistic quantum mechanics, we write the Dirac equation as:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{10}$$

where the 4×4 gamma matrices γ^μ can be written ("Bjorken and Drell" convention):

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and σ are the Pauli spin matrices.

Since $\psi^\dagger\psi$ is not a relativistic invariant, it is convenient to work with $\overline{\psi}\psi$, where $\overline{\psi} \equiv \psi^\dagger\gamma^0$. It can be shown that $\overline{\psi}\psi$ is a relativistic invariant, and that the "bilinears" transform as follows:

1. $\overline{\psi}\psi =$ scalar,
2. $\overline{\psi}\gamma^5\psi =$ pseudo-scalar,
3. $\overline{\psi}\gamma^\mu\psi =$ vector,
4. $\overline{\psi}\gamma^\mu\gamma^5\psi =$ pseudo-vector,
5. $\overline{\psi}\sigma^{\mu\nu}\psi =$ antisymmetric tensor,

where $\sigma^{\mu\nu} \equiv i/2[\gamma^\mu, \gamma^\nu]$.

The free Lagrange density for fermions is written

$$\mathcal{L}_0 = \overline{\psi}(i\partial^\mu\gamma_\mu - m)\psi. \tag{11}$$

To conserve parity in the interaction Lagrange-density for a pseudo-scalar particle interacting with a fermion spinor field (in this case we will only consider proton fields), one needs to use a γ^5 factor so that,

$$\mathcal{L}_{int} = -g\overline{\psi}\gamma^5\psi\phi. \tag{12}$$

The full Lagrange density can now be written

$$\mathcal{L} = \overline{\psi}[i\partial^\mu\gamma_\mu - m_p]\psi + \frac{1}{2}[(\partial\phi)^2 - m_\pi^2\phi^2] - g\overline{\psi}\gamma^5\psi\phi,$$

and is invariant under a parity transformation.

Without going into the technical details (a full derivation can be found in section 9.9 of "Relativistic Quantum Mechanics and Field Theory" by Franz Gross) the resulting non-relativistic potential becomes:

$$V(r) = \frac{g^2 m_\pi^2}{48\pi m_p^2}\left[\frac{e^{-m_\pi r}}{r}\right] \times \{\vec{\sigma_1}\cdot\vec{\sigma_2} +$$
$$(3\vec{\sigma_1}\cdot\hat{r}\vec{\sigma_2}\cdot\hat{r} - \vec{\sigma_1}\cdot\vec{\sigma_2})\left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right)\Bigg\}. \tag{13}$$

What a difference a γ^5 makes! At first glance, this seems to be a disaster, but if we look closer, it turns out not to be all that bad. For one, we see the common factor of $e^{-m_c r}/r$. Also, we can see terms:

- $(\vec{\sigma_1}\cdot\vec{\sigma_2})$: central spin dependence,
- $(3\vec{\sigma_1}\cdot\hat{r}\vec{\sigma_2}\cdot\hat{r} - \vec{\sigma_1}\cdot\vec{\sigma_2})$: tensor,

that are needed for a description of nucleon-nucleon interactions ☺. Unfortunately, the central spin dependence is about a factor of ten smaller than what is found in NN scattering and there is a spin-orbit part that is completely missing ☹. The model can, however, be improved upon by allowing scalar and vector mesons.

This is a cautionary tale: What may appear to be a simple change to the vertex can result in large changes in the behavior of the reaction.

This is a good opportunity to talk about recoil polarimeters. If our model included vector meson exchange, we would be able to generate another term to our potential that couples spin and orbital angular momentum [1].

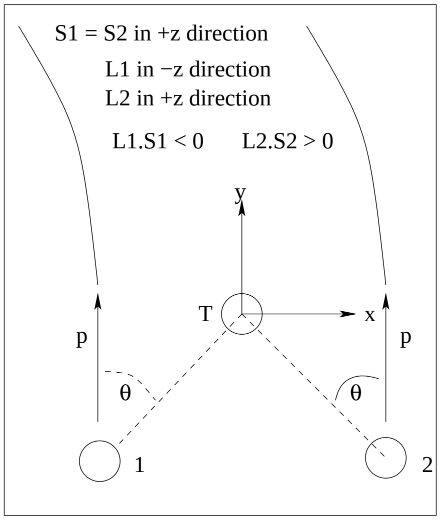

FIG. 4: Spin orbit coupling.

To get an intuitive idea as to why we should expect a spin-1 vector meson to give a spin-orbit interaction term, we will work by analogy. We know that the photon is a spin-1 massless field, and that the coupling of the photon field to fermions must produce spin-orbit interactions. (I say this with confidence because we know that spin-orbit interactions have been measured in atomic spectroscopy.) So, if we were to couple a massless spin-1 vector meson to a fermion field, we should obtain a spin-orbit coupling. The only difference between the spin-orbit coupling for a massless spin-1 meson, and a photon, should come from the different strengths of the coupling constants. In our case, we are interested in a massive meson field. This means that the propagator for the massive field is expected to cause a dampening effect in the potential. Otherwise, we should expect the two fields to act similarly (more on this later on).

We will consider a spin orbit term of the form:

$$V_{SO} = -C\vec{L}\cdot\vec{S},$$

where C is a positive constant.

If we had a target of carbon atoms (which is often the case for a recoil polarimeter), and incident protons as shown in Fig. 4, where the spin of each incident proton is aligned out of the page, we would obtain a repulsive potential for proton 1 and an attractive potential for proton 2. For each case where the proton spin is directed out of the page, we get a force bending the path to the left! If the spin were to be pointed into the page, instead, the paths would be bent to the right. This is how a recoil polarimeter works: a slab of carbon is placed in a detector, and the direction and degree of deflection tells us about the proton spin polarization.

VIII. THE QED LAGRANGIAN DENSITY

For a photon field we have the free Lagrange density

$$\mathcal{L}_0 = \frac{-1}{4}F^{\mu\nu}F_{\mu\nu},$$

where

$$\begin{aligned} F^{\mu\nu} &\equiv \partial^\mu A^\nu - \partial^\nu A^\mu \\ &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \end{aligned}$$

implies that

$$\frac{-1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2). \tag{14}$$

This is just the difference between the energy density of the electric and magnetic fields. So when you see $-F^{\mu\nu}F_{\mu\nu}/4$ you should think $(\vec{E}^2 - \vec{B}^2)/2$.

Now that the Lagrange density for a free photon field has an intuitive deffinition, we want to see how the photon field is coupled to a fermion field. Earlier we saw that the free Lagrangian for a fermion field is written as

$$\mathcal{L}_0 = \overline{\psi}(i\partial^\mu\gamma_\mu - m)\psi.$$

The interaction part for the two fields ψ, A_μ will be

$$\mathcal{L}_{int} = -e\overline{\psi}\gamma^\mu\psi A_\mu.$$

(To make sense of this $\mathcal{L}_{int}$ term you should notice that we want to couple a four-vector electromagnetic field A_μ to the fermion field forming a scalar [recall that the Lagrange density must be a scalar]. The only way to do this is to contract the four-vector A_μ with some other four-vector. The only vector bilinear at our disposal is $\overline{\psi}\gamma^\mu\psi$.) Putting this all together we have

$$\mathcal{L}_{QED} = \overline{\psi}(i\partial^\mu\gamma_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\overline{\psi}\gamma^\mu\psi A_\mu. \tag{15}$$

Often you will see this written as

$$\mathcal{L}_{QED} = \overline{\psi}[i\gamma_\mu D^\mu - m]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

where $D^\mu \equiv \partial^\mu + ieA_\mu$. These two forms given for $\mathcal{L}_{QED}$ are identical, but look different at first glance.

The interaction vertex has a Feynman diagram as shown in Fig. 5, and the vertex factor can be read directly from the interaction part of $\mathcal{L}_{QED}$ and is $-e\gamma_\mu$. This is the only vertex possible, as can be seen from the Lagrange density in Eqn. 15. Every Feynman diagram you can think of for QED is made up of this one vertex. For example we can get a diagram for photon scattering $\gamma\gamma \to \gamma\gamma$ as shown in Fig. 6 just by using our one simple vertex over and over.

The propagators for this theory are

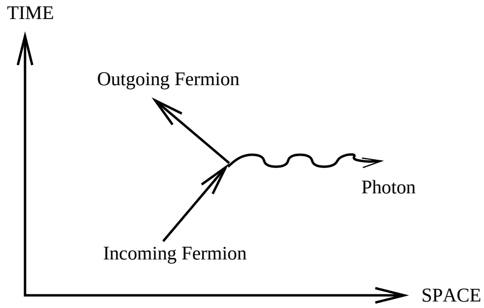

FIG. 5: Vertex for QED.

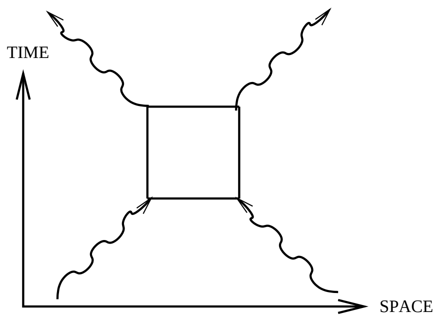

FIG. 6: A Feynman diagram for $\gamma\gamma \to \gamma\gamma$.

- photons: $-ig^{\mu\nu}/q^2$,
- fermions: $i/(\gamma^\mu q_\mu - m)$,

and are derivable from the free Lagrange densities. As would be expected, the non-relativistic potential is $V \propto 1/r$ for unpolarized fermions, and depends on the photon mass being zero.

To distinguish between particles and anti-particles within a Feynman diagram, the time-direction of the arrow is used: Particles will have arrows pointing in the positive time direction, while anti-particles will point in the negative time direction. For example, Fig. 7 shows $e+\bar{\mu} \to e+\bar{\mu}$, where $\bar{\mu}$ represents the μ anti-particle, and Fig. 8 shows one diagram for the pair annihilation process $e^- + e^+ \to \gamma + \gamma$. (The anti-particle for e is denoted e^+ instead of as $\bar{e}$ for purely historical reasons.)

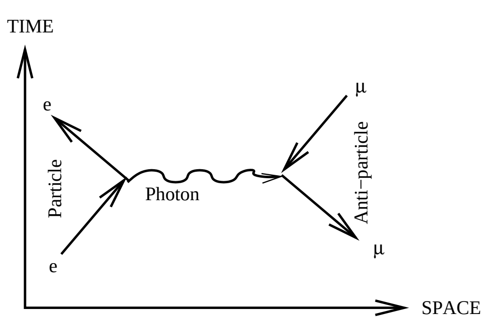

FIG. 7: A Feynman diagram for $e + \bar{\mu} \to e + \bar{\mu}$, where the direction of the arrows determines the particle, anti-paritcle nature.

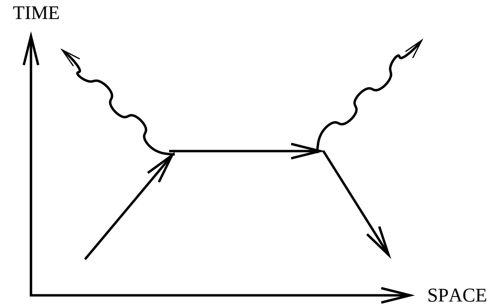

FIG. 8: A Feynman diagram for $e^- + e^+ \to \gamma + \gamma$.

For identical particles in the final state, we can not distinguish between the Feynman diagram given in Fig. 8 and that shown in Fig. 9. This means that both figures would need to be used in determining the $\mathcal{M}$-matrix for the second order $e^- + e^+ \to \gamma + \gamma$ process.

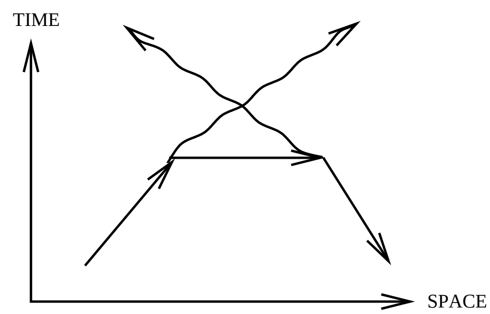

FIG. 9: A second Feynman diagram for $e^- + e^+ \to \gamma + \gamma$.

One last issue before we move on to the next subsection. We need to keep track of the spin of the fermion lines and the polarization of the photon lines. In Feynman diagrams, you associate a spinor for each external fermion line and a polarization vector for each external photon line.

Most particle physics books will give you the Feynman rules for QED, and show you how to calculate the $\mathcal{M}$-matrix for specific processes.

A. Anomalous magnetic moment and effective Lagrange densities

Since we can always make more complicated Feynman diagrams, it would be good make modifications to the rules so that, while the Lagrange density has not changed, the modified rules "soak up" some common classes of Feynman diagrams. One such modification is associated with something called the anomalous magnetic moment. This anomalous magnetic moment results from a class of Feynman diagrams where there are photon lines "straddling" a vertex. An example can be found in Fig. 10 and the sum of all of these type of corrections is commonly drawn as in Fig. 11.

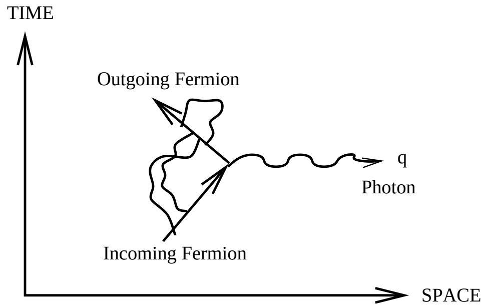

FIG. 10: A Feynman diagram for a vertex correction.

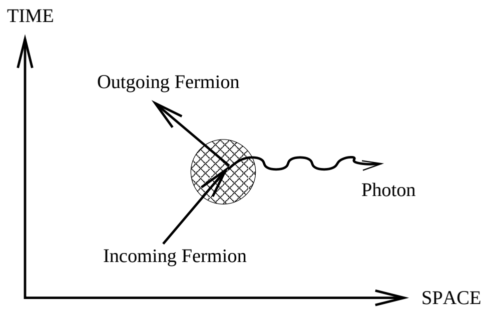

FIG. 11: A Feynman diagram for all possible vertex corrections.

We can make a simple change to our vertex function that accounts for the anomalous magnetic moment by changing our vertex function to be

$$\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_N} F_2(q^2),$$

where F_1 and F_2 are functions of momentum-transfer that are determined for the particular fermion fields involved, and m_N is the proton mass. While the fundamental Lagrange density has not changed, we could write down an effective interaction Lagrange density such that

$$\mathcal{L}_{int\ \mathrm{Effective}} = -F_1(q^2)\overline{\psi}\gamma^\mu\psi A_\mu + F_2(q^2)\overline{\psi}\left(\frac{\sigma^{\mu\nu}}{2m_N}\right)\psi F_{\mu\nu},$$

where it is understood that the class of Feynman diagrams as shown in Fig. 10 are already taken care of.

It might seem a bit strange to replace $i\sigma^{\mu\nu}q_\nu A_\mu$ with $\sigma^{\mu\nu}F_{\mu\nu}/2$. To make sense of this substitution, you need to know that the A^μ field can be written as

$$A^\mu = \sum_s \int \frac{d^4q\,\epsilon_s^\mu}{(2\pi)^3\sqrt{E}}\left(a_{s\vec{q}} + a^\dagger_{s-\vec{q}}\right)\, e^{-ix\cdot q},$$

where ϵ^μ is the polarization vector, and implies that

$$\begin{aligned}\partial^\nu A^\mu &= i\sum_s \int \frac{d^4q\,q^\nu\epsilon_s^\mu}{(2\pi)^3\sqrt{E}}\left(a_{s\vec{q}} + a^\dagger_{s-\vec{q}}\right)\, e^{-ix\cdot q}.\\ &= -iq^\nu A^\mu.\end{aligned}$$

You also need to recognize that since $\sigma^{\mu\nu}$ is antisymmetric under $\mu \to \nu$:

$$\begin{aligned}\sigma^{\mu\nu}q_\nu A_\mu &= -\sigma^{\mu\nu}q_\mu A_\nu \Rightarrow\\ &= \frac{1}{2}(\sigma^{\mu\nu}q_\nu A_\mu - \sigma^{\mu\nu}q_\mu A_\nu)\\ &= \frac{-i}{2}\sigma^{\mu\nu}F_{\mu\nu}.\end{aligned}$$

Thus, $i\sigma^{\mu\nu}q_\nu A_\mu = \sigma^{\mu\nu}F_{\mu\nu}/2$.

What we will want to know next is how massive spin-1 mesons (e.g. ρ, ω, and ϕ) can be incorporated into a model for nucleon interactions.

IX. MASSIVE SPIN-1 VECTOR FIELD

As a reminder, we can write the free Lagrange density for a photon field as

$$\mathcal{L}_0 = \frac{-1}{4}F^{\mu\nu}F_{\mu\nu},$$

where

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu.$$

The most general Lagrange density we could write for a gauge field and fermion field is

$$\mathcal{L} = \overline{\psi}[i\gamma_\mu D^\mu - m]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - c\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}. \quad (16)$$

(see section 15.1 of "An Introduction to Quantum Field Theory", by M.E. Peskin and D.V. Schroeder.) The term $\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$ violates parity and is thrown out. We will come back to this equation (Eqn. 16) later on.

For a spin-1 massive vector field we set

$$\mathcal{L}_0 = \frac{-1}{4}V^{\mu\nu}V_{\mu\nu} - \frac{1}{2}m_V^2 V^\mu V_\mu,$$

where

$$V^{\mu\nu} \equiv \partial^\mu V^\nu - \partial^\nu V^\mu,$$

V^μ represents the massive vector field, and m_V is the mass of the particle excitation of the vector field V^μ. The interaction term for our massive vector field coupled to a fermion field is given as

$$\begin{aligned}&\mathcal{L}_{int} = -\lambda\overline{\psi}\gamma^\mu\psi V_\mu,\\ &\mathcal{L}_{int\ \mathrm{Effective}} =\\ &-G_1(q^2)\overline{\psi}\gamma^\mu\psi V_\mu + G_2(q^2)\overline{\psi}\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m_N}\right)\psi V_{\mu\nu},\end{aligned}$$

where G_1, G_2 are analogs of the form factor F_1, F_2. Immediately, we see that if we take $m_V = 0$, then the only difference between the Lagrange density for our meson vector-field and the photon field, is the coupling constants. (For the effective Lagrangian we also see different form factors.) Also, the transformation properties of

$F^{\mu\nu}$ will be identical to the transformation properties of $V^{\mu\nu}$, and thus there are quantities within $V^{\mu\nu}$ that are a direct analog to the $\vec{E}$ and $\vec{B}$ fields of electromagnetism.

Because of the mass term in the free Lagrange density, the propagator for massive vector field will be different than that for the massless photon field ($-ig^{\mu\nu}/q^2$), and is

$$\frac{-i(g^{\mu\nu} - q^\mu q^\nu/m_V^2)}{q^2 - m_V^2}. \tag{17}$$

The first thing you should notice is that the propagator given in Eqn. 17 does not behave nicely in the limit that $m_V \to 0$. What we would like to see, is the massive propagator, in the limit $m_V \to 0$, to become the photon propagator $-ig^{\mu\nu}/q^2$. The reason that the two propagators look so different has to do with the fact that a massless vector field has only two polarization degrees of freedom.

A technical detail: For an arbitrary spin-1 particle we can have three polarization states: One longitudinal state and two transverse states. For a massless particle, the velocity must be the speed of light. This causes the length contraction in the direction of motion to completely "wipe-out" any vector quantity in that direction. This is why a photon is spin-1 with only two polarizations. The photon field has "lost" a degree of freedom. Because of this lost degree of freedom, we have a field that is under-determined. This manifests itself in the photon field having a non-determined "gauge". We have the freedom to choose an electromagnetic gauge transformation such that

$$A^\mu \to A^\mu + \partial^\mu f(\vec{x}, t), \tag{18}$$

where f is any function of position and time. This choice of gauge transformation manifests itself in the photon propagator such that the propagator can be written as

$$\frac{-i}{q^2}\left[g^{\mu\nu} - (1-\zeta)\frac{q^\mu q^\nu}{q^2}\right], \tag{19}$$

where ζ can be any number and represents a choice of gauge. In this document $\zeta = 1$ is chosen so that the photon propagator becomes $-ig^{\mu\nu}/q^2$. I still have not shown how the massive vector-field propagator becomes the massless vector propagator, but have instead shown that determining the massless vector propagator is a tricky affair (that we won't go into any further).

X. EFFECTIVE LAGRANGIAN THEORY FOR η PHOTOPRODUCTION

I will now briefly describe the interaction parts of an effective field theory by Benmerrouche, Mukhopadhyay, and Zhang (BMZ). These interaction Lagrange densities are also common to the theory of Nakayama and Haberzettl (NH).

I will use the notation of BMZ:

- $N \to$ Proton field.
- $\eta \to \eta$ field.
- $V \to$ Vector meson field (ρ or ω).
- $m_N \to$ Proton mass.
- $A^\mu \to$ Photon field.
- $R \to$ Nucleon resonance field.

A. Interaction Lagrange density for γNN

We have already worked out the effective Lagrange density for a photon-field interacting with a fermion field. As previously discussed:

$$\mathcal{L}_{\gamma NN} = -e\overline{N}\gamma_\mu N A^\mu + \frac{\kappa_t}{2m_N}\overline{N}\sigma_{\mu\nu}NF^{\mu\nu}.$$

In this case $F_1 = e$, and $F_2 = \kappa_t$.

B. Interaction Lagrange density for VNN

We have also shown that for a vector-field coupled to a fermi-field, we expect (by direct analogy to the photon case):

$$\mathcal{L}_{VNN} = -g_v\overline{N}\gamma_\mu N A^\mu + \frac{g_t}{2m_N}\overline{N}\sigma_{\mu\nu}NF^{\mu\nu}.$$

In this case $G_1 = g_v$, and $G_2 = g_t$.

C. Interaction Lagrange density for ηNN

The Lagrange density for ηNN has an additional pseudo-vector coupling term we have not discussed thus far. This new pseudo-vector coupling can be seen in the second term of the Lagrange density

$$\mathcal{L}_{\eta NN} = g_\eta\left[-i\zeta\overline{N}\gamma_5 N\eta + (1-\zeta)\frac{1}{2}\overline{N}\gamma_\mu\gamma_5 N\partial^\mu\eta\right],$$

where ζ represents the fraction of the pseudo-scalar to pseudo-vector couplings. Previously, when discussing the anomalous magnetic moment, we found that the effective Lagrange density can be described using field-derivative couplings. It turns out that the general form for our Lagrange density for ηNN can include an η-field derivative. In the case of πNN, the pseudo-vector piece dominates to the extent that the pseudo-scalar piece is often left out entirely ($\zeta \to 0$) for pion-nucleon-nucleon interactions.

D. Interaction Lagrange density for ηNR

These terms are as expected, once the parity of the resonance state R is taken account of:

$$\begin{aligned}
\mathcal{L}^{PS}_{\eta NR} &= -ig_{\eta NR}\overline{N}\Gamma R\eta \\
\mathcal{L}^{PV}_{\eta NR} &= -i\frac{f_{\eta NR}}{m_\eta}\overline{N}\Gamma_\mu R\partial^\mu\eta, \\
\mathcal{L}_{\gamma NR} &= \frac{e\kappa}{2(m_R+m_N)}\overline{R}\,\Gamma_{\mu\nu}NF^{\mu\nu},
\end{aligned}$$

where

- Odd parity $\Rightarrow \Gamma = 1$, $\Gamma_\mu = \gamma_\mu$, $\Gamma_{\mu\nu} = \gamma_5\sigma_{\mu\nu}$
- Even parity $\Rightarrow \Gamma = \gamma_5$, $\Gamma_\mu = \gamma_\mu\gamma_5$, $\Gamma_{\mu\nu} = \sigma_{\mu\nu}$,
- Super script $PV \Rightarrow$ pseudo-vector coupling,
- Super script $PS \Rightarrow$ pseudo-scalar coupling.

We can readily see that the only difference between the odd- and even-resonance cases is a factor of γ_5 to keep the overall parity to be even.

E. Interaction Lagrange density for $\eta\gamma V$

We want to include a term that couples the η-field to the γ-field and vector meson V-field. As usual, the fact that the η-field is pseudo-scalar means that we have to be careful about parity. In our discussion regarding the most general form for the Lagrange density for a Dirac field coupled to a vector gauge field seen in Eqn. 16, I stated that the term $\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$ was odd under a parity transformation.

To easily see the parity properties of $F^{\mu\nu}$, it is useful to look at the individual components. The electromagnetic A^μ field transforms like a regular vector, and since

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t}, \\
\vec{B} &= \vec{\nabla}\times\vec{A},
\end{aligned}$$

then $\vec{E}$ is a vector and $\vec{B}$ is a pseudo-vector. Another way to think about the magnetic transformation properties is to imagine an infinite positive line charge moving in the $+z$ direction. The magnetic field for this moving line charge will have a magnetic field pointed in the $+\hat{\phi}$-direction (taking cylindrical coordinates). If we were to invert space ($\vec{r}\to-\vec{r}$), then the direction of the $\vec{B}$-field remains in the $+\hat{\phi}$-direction. Since $F^{\mu\nu}$ can be made from E and B components, then $F^{\mu\nu}$ is of mixed parity. The combination $-F^{\mu\nu}F_{\mu\nu}/4 = (\vec{E}^2-\vec{B}^2)/2$, however, is even under a parity transformation.

Since the strong and electromagnetic interactions conserve parity, then a term like $\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$ must be excluded in the QED Lagrangian. Now, however, what was a fatal flaw for including this type of term for QED, is an asset when considering an interaction term for coupling η to V and γ. It is now reasonable to take the form of $\mathcal{L}_{V\eta\gamma}$ to be:

$$\mathcal{L}_{V\eta\gamma} = \frac{e\lambda_V}{4m_\eta}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}V^{\alpha\beta}\eta.$$

As discussed previously, $V^{\mu\nu}$ has the same transform properties as $F^{\mu\nu}$. This means that there will be quantities within $V^{\mu\nu}$ that act as direct analogs to the electromagnetic vectors $\vec{E}$, and $\vec{B}$. Let me write these analogs as $\underline{\vec{E}}$, and $\underline{\vec{B}}$ so that

$$\begin{aligned}
V^{\mu\nu} &\equiv \partial^\mu V^\nu - \partial^\nu V^\mu \\
&= \begin{pmatrix} 0 & -\underline{E}_x & -\underline{E}_y & -\underline{E}_z \\ \underline{E}_x & 0 & -\underline{B}_z & \underline{B}_y \\ \underline{E}_y & \underline{B}_z & 0 & -\underline{B}_x \\ \underline{E}_z & -\underline{B}_y & \underline{B}_x & 0 \end{pmatrix},
\end{aligned}$$

and

$$\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}V^{\alpha\beta} = \vec{E}\cdot\underline{\vec{B}} + \underline{\vec{E}}\cdot\vec{B}. \tag{20}$$

We see that the electromagnetic $\vec{E}$-field couples to the vector-analog $\underline{\vec{B}}$-field, and the electromagnetic $\vec{B}$-field couples to the vector-analog $\underline{\vec{E}}$-field. Moreover, since $\vec{E}$ and $\underline{\vec{E}}$ are vector, and $\vec{B}$ and $\underline{\vec{B}}$ are pseudo-vector, then $\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}V^{\alpha\beta} = \vec{E}\cdot\underline{\vec{B}} + \underline{\vec{E}}\cdot\vec{B}$ will be a pseudo-scalar as we desire for coupling to the pseudo-scalar η meson.

F. Contact term $NN\gamma\eta$

To maintain the photon-field gauge invariance, a contact term is required. The sole purpose of this $NN\gamma\eta$ interaction is to subtract out bad (gauge violating) behavior. You can think of it as an Ad Hoc addition to the model to keep it from falling apart.

G. Resonances with spin 3/2 ($R_{3/2}$)

The NH model includes resonances with spin 3/2. This extra degree of freedom for the resonance mediator adds additional complications. The Lagrange density for this spin 3/2 case we have

$$\mathcal{L}_{NR_{3/2}\eta} = \frac{g_{NR_{3/2}\eta}}{m_\eta}\overline{R}^\mu\Theta_{\mu\nu}(z)\Gamma^{(\pm)}N\partial\eta,$$

where $\Gamma(+) = \gamma_5$, $\Gamma(+) = 1$, and $\Theta_{\mu\nu}(z) = g_{\mu\nu} - (z - 1/2)\gamma_{mu}\gamma_\nu$.

We also have a modification to the $\gamma R_{3/2}N$ coupling:

$$\begin{aligned}
\mathcal{L}_{\gamma R_{3/2}N} = i\ \ & e\frac{g_{1\gamma R_{3/2}N}}{m_{R_{3/2}}}\overline{R}_\mu\gamma_5\gamma_\nu NF^{\mu\nu} + \\
& e\frac{g_{2\gamma R_{3/2}N}}{m_{R_{3/2}}}\partial_\nu\overline{R}_\mu\gamma_5 NF^{\mu\nu}.
\end{aligned}$$

Since I am not comfortable talking about fields with spin $\geq 3/2$, I'll have to end the discussion of $R_{3/2}$ here ☹.

It has taken some time, but now we have shown all of the Lagrange densities associated with the BMZ and NH models for η and η' photoproduction ☺ Now we want to see how one of these models compares to actual data.

XI. RESULTS OF THE NH MODEL APPLIED TO ASU η' PHOTOPRODUCTION DATA

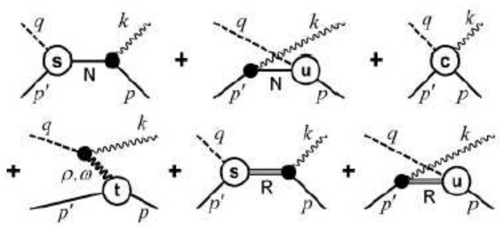

FIG. 12: Feynman diagrams for the NH model. The variable q is associated with the η', k the incident photon, p the target proton, p' the recoil proton, R the resonance, and ρ/ω the vector mesons.

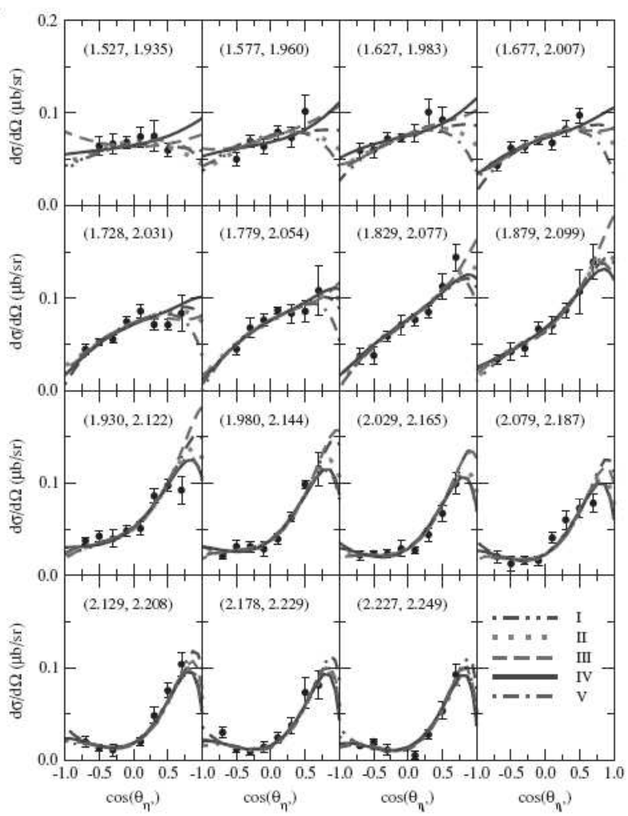

FIG. 13: NH model differential cross sections for five different sets of assumed resonance contributions (I through V).

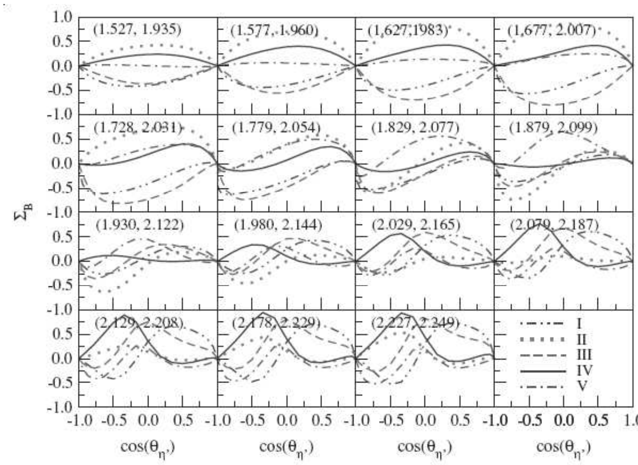

FIG. 14: NH model beam asymmetries for the five different sets of assumed resonance contributions (I through V).

The NH effective Lagrangian model has the Feynman diagrams as shown in Fig. 12

When all of the production mechanisms shown in Fig. 12 are added together for five different sets of assumed resonance contributions, NH obtain the fits to the ASU differential cross sections shown in Fig. 13.

We can see that each of these five sets (I through V) compare well with the ASU results. This means that we will need another type of observable to help deconvolute the spectrum. One such observable is the beam asymmetry. A plot of the beam asymmetry from the NH model for the five sets of resonances can be found in Fig. 14.

It is easy to see from Fig. 14 that the beam asymmetries should help greatly in determining the best possible set of included resonances for η' photoproduction. A former member of the ASU group, Patrick Collins, is currently working on the beam asymmetry measurements for this reaction.

XII. CONCLUSION

We have discussed interaction Lagrange densities for photoproduction of η and η' (from the proton) commonly found in the literature regarding these reactions. We have also seen how the results of the NH model compare to ASU η' photoproduction data.

XIII. ACKNOWLEDGMENTS

I thank Barry Ritchie and William Kaufmann for their comments.

This work was supported at Arizona State University by the National Science Foundation award PHY-0653630.

[1] G. Breit, Phys. Rev. **120**, 287 (1960)

APPENDIX: FINDING THE Z AMPLITUDE

There are two common methods for finding the $\mathcal{M}$-matrix: (1) Canonical method using creation and annihilation operators; (2) Path integrals. Each of these approaches has strengths and weaknesses. One strength of the path integral approach is that the relation of $\mathcal{M}$-matrix to the Lagrange density is more clearly seen than in the canonical formalism.

Most field theory text include a description of both formalisms. It is probably more common for the canonical treatment to be emphasized initially (e.g. "An Introduction to Quantum Field Theory" by Michael E. Peskin, and Daniel V. Schroeder; "Quantum Field Theory", by Michio Kaku; "Relativistic Quantum Mechanics and Field Theory", by Franz Gross; "The Quantum Theory of Fields, Vol 1", by Stephen Weinberg . . .). Most likely, this is because the canonical approach looks more like the quantum mechanics you have encountered as undergraduates, with operators acting on quantum-mechanical bra's and ket's. However, some texts will start by emphasizing the path integral, with one such example being "Quantum Field Theory in a Nutshell", by A. Zee. In what follows, I have chosen to discuss the path integral approach of determining Z.

It is not the intent of this appendix to show how the path integral is put together from first principles. For a complete treatment of the path integral formalism you might want to read Zee. I will, instead, state a few results, so as to more clearly show why I constructed the Z amplitudes in the manner that I have shown in the document.

I start by simply stating the the time development of a state out-of and into the vacuum can be written as

$$Z \equiv \langle 0|e^{-iHT}|0\rangle = \int D\phi e^{iS}, \qquad (21)$$

where H is the Hamiltonian, $\int D\phi \Rightarrow$ integration over all possible paths, and the action $S = \int dtL = \int d^4x\mathcal{L}$.

Since we are constructing states that must come from the vacuum at some space-time points and then returning to the vacuum at other space-time points (we have constructed the Z-amplitude so that the initial and final states are the vacuum), we need a mechanism that acts as sources and sinks for our "external" particles within any given Feynman diagram. To accomplish this, we add terms to the Lagrange density: $J\phi$, where ϕ represents the field of our external line and J represents sources/sinks. Fortunately, we can take the sources and sinks to be space-time delta functions $J = \delta^{(4)}(x - x_1)$, where x_1 represents the space-time point that a field ϕ is "pulled out" of the vacuum (source), or "put into" the vacuum (sink).

The amplitude Z can now be seen as a function of the sources and sinks, and by noticing that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} + J\phi$, we can write

$$Z = \int D\phi e^{i\int(d^4x\mathcal{L}_0+\mathcal{L}_{int}+J\phi)}.$$

We are almost done! We are simply going to expand Z in terms of J, and then expand Z in terms of the interaction to get:

$$\begin{aligned} Z &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} Z(j,k) = \int D\phi e^{i\int d^4x\mathcal{L}_0} \times \\ &\sum_{j=1}^{\infty}\frac{1}{j!}\left[i\int d^4xJ\phi\right]^j \sum_{k=1}^{\infty}\frac{1}{k!}\left[i\int d^4x\mathcal{L}_{int}\right]^k. \end{aligned}$$

Let us see what we get when we take $J = \delta^{(4)}(x-a) + \delta^{(4)}(x-b) + \delta^{(4)}(x-c)$, and $\mathcal{L}_{int} = -g\phi^3/(3!)$ to the order $j = 3$, and $k = 1$:

$$\begin{aligned} Z(3,1) &= -i\frac{g}{3!}\int d^4x \int D\phi e^{i\int d^4x\mathcal{L}_0}\phi(x)\phi(x)\phi(x) \times \\ &\frac{1}{3!}\left[i\int d^4xJ\phi\right]^3. \end{aligned}$$

The first thing that we notice is that the terms in J are going to give us a mess!

We can fix the terms in J by noticing that combination like $[\delta^{(4)}(x-a)]^3$, and $[\delta^{(4)}(x-a)]^2\delta^{(4)}(x-b)$ will describe a process where one or more of our external lines share the same space-time point of origin or "death". We are not interested in processes that have external lines that share a common beginning or end, so we only consider the term $3!\,\delta^{(4)}(x-a)\delta^{(4)}(x-b)\delta^{(4)}(x-c)$. Now we have

$$\begin{aligned} Z(3,1) &= -\frac{(i)^4}{3!}\int d^4x \int D\phi e^{i\int d^4x\mathcal{L}_0} \times \\ &\phi(x)\phi(x)\phi(x)\phi(a)\phi(b)\phi(c). \end{aligned} \qquad (22)$$

To evaluate the integral in Eqn. 22 we need an identity that I will state without proof:

$$\begin{aligned} &\frac{(i)^3}{Z(0,0)}\int D\phi e^{i\int d^4x\mathcal{L}_0} \times \phi(x)\phi(x)\phi(x)\phi(a)\phi(b)\phi(c) \\ &= D(a-x)D(b-x)D(c-x) + D(a-x)D(c-x)D(b-x) \\ &+ D(b-x)D(c-x)D(a-x) + D(b-x)D(c-x)D(a-x) \\ &+ D(c-x)D(b-x)D(a-x) + D(c-x)D(a-x)D(b-x) \\ &+ D(x-x)D(a-b)D(c-x) + \cdots, \end{aligned} \qquad (23)$$

where I have connected field pairs in every possible way to construct the propagators.

The first 3! (six) terms of Eqn. 23 can be represented by the single Feynman diagram shown in Fig. 15. Whereas, a term like $D(x-x)D(a-b)D(c-x)$ will yield a "disconnected" diagram as shown in Fig. 16.

We are not interested in the disconnected diagrams, so we can now write

$$\begin{aligned} Z &= \frac{Z(3,1)}{Z(0,0)} = \\ &-ig\int d^4xD(a-x)D(b-x)D(c-x). \end{aligned}$$

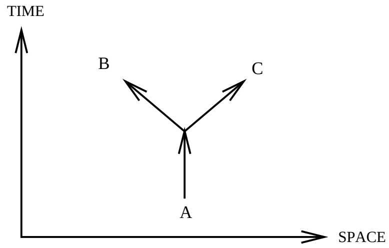

FIG. 15: A Feynman diagram for the first 3! (six) terms in Eqn. 23.

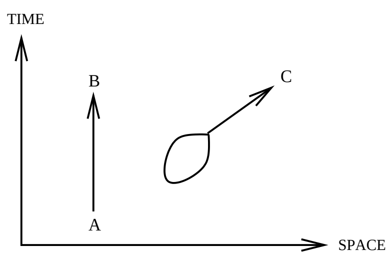

FIG. 16: A Feynman diagram for $D(x - x)D(a - b)D(c - x)$.