Anti-sky mapping
& Lorentz transforms
of circular silhouette's

Two frames (S & S') with anti-sky-mapping in frame S

Frame S $\left\{\begin{array}{l} x_s = \sin\theta\cos\varphi \\ y_s = \sin\theta\sin\varphi \\ z_s = \cos\theta \\ ct_s = 1 \end{array}\right\}$ Boost in z-direction $\longrightarrow$ $\left\{\begin{array}{l} x' = \sin\theta\cos\varphi \\ y' = \sin\theta\sin\varphi \\ z' = \gamma\cos\theta - \gamma\beta \\ ct' = \gamma - \gamma\beta\cos\theta \end{array}\right\}$ Frame S'

The light rays in S' are not all on unit sphere. To get rays on unit sphere, we just need to divide x', y', z', ct' by

$ct' = \gamma - \gamma\beta\cos\theta$ {hypotenuse}

$$\Rightarrow \quad x'_s = \frac{\sin\theta\cos\varphi}{\gamma - \gamma\beta\cos\theta}, \quad y'_s = \frac{\sin\theta\sin\varphi}{\gamma - \gamma\beta\cos\theta}, \quad z'_s = \frac{\gamma\cos\theta - \gamma\beta}{\gamma - \gamma\beta\cos\theta}$$

& $ct' = 1$ Now S' is anti-sky-mapped ☺

<u>special case</u> $\quad z'^2 + x'^2 = 1 \quad$ {i.e. $\varphi = 0 \Rightarrow y' = 0$}

Now $$1 = \frac{\gamma^2(\cos\theta - \beta)^2}{\gamma^2(1-\beta\cos\theta)^2} + \frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-\beta\cos\theta)^2} \Rightarrow$$

$$(1-\beta\cos\theta)^2 = (\cos\theta - \beta)^2 + \sin^2\theta\cos^2\varphi/\gamma^2 \Rightarrow$$

$$1 - 2\beta\cos\theta + \beta^2\cos^2\theta = \cos^2\theta - 2\beta\cos\theta + \beta^2 + \sin^2\theta\cos^2\varphi(1-\beta^2)$$

$$1 = (1-\beta^2)\cos^2\theta + \beta^2 + (1-\beta^2)\sin^2\theta\cos^2\varphi$$

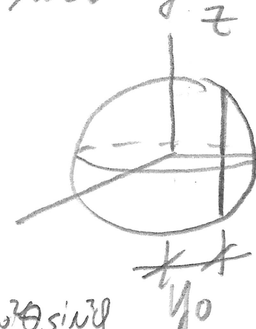
$$(1-\beta^2) = (1-\beta^2)(\cos^2\theta + \sin^2\theta\cos^2\varphi) \qquad 1 = \cos^2\theta + \sin^2\theta\cos^2\varphi$$

$\Rightarrow \quad 1 = x^2 + z^2 \quad$ on plane $y = 0$

So, for this special case

an anti-sky-mapped circle Lorentz transforms to an anti-sky-mapped circle ☺

Now, move off the meridian by moving circle from $y=0$ to $y=y_0$.



Our equation becomes $z'^2+x'^2+y_0'^2=1$

$\Rightarrow$

$$1 = \frac{\gamma^2(\cos\theta-\beta)^2}{\gamma^2(1-\beta\cos\theta)^2} + \frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-\beta\cos\theta)^2} + \frac{\sin^2\theta\sin^2\varphi}{\gamma^2(1-\beta\cos\theta)^2}$$

$$\Rightarrow (1-\beta\cos\theta)^2 = (\cos\theta-\beta)^2 + \sin^2\theta\cos^2\varphi/\gamma^2 + \sin^2\theta\sin^2\varphi/\gamma^2$$

$$\Rightarrow 1-2\beta z+\beta^2 z^2 = z^2-2\beta z+\beta^2 + x^2/\gamma^2 + y^2/\gamma^2$$

$$\Rightarrow (1-\beta^2) = z^2(1-\beta^2) + x^2(1-\beta^2) + y^2(1-\beta^2)$$

$$\Rightarrow 1 = z^2+x^2+y^2$$ As it must

In S' the plane is defined as

$$y = y_0 \Rightarrow y_0' = \frac{\sin\theta\sin\varphi}{\gamma(1-\beta\cos\theta)} \Rightarrow$$

$$y_0' = \frac{y}{\gamma-\gamma\beta z} \Rightarrow y_0'\gamma - \gamma\beta y_0' z = y \Rightarrow y+\gamma\beta y_0' z = y_0'\gamma$$

$\Rightarrow \underbrace{ay+bz=p}_{\text{Equation of a plane parallel to x-axis}}$ where $a=1$, $b=\gamma\beta y_0'$ & $p=y_0'\gamma$

Don't need to show that the points map onto unit sphere in S any more.

Just need to make sure that the points map onto a plane in S

General case $\rightarrow$

$l'x' + m'y' + n'z' = p'$ → Eq$^{\underline{n}}$ of plane in S'

$$\Rightarrow \quad l'\frac{\sin\theta\cos\phi}{\gamma-\gamma\beta\cos\theta} + m'\frac{\sin\theta\sin\phi}{\gamma-\gamma\beta\cos\theta} + n'\frac{(\gamma\cos\theta-\gamma\beta)}{\gamma-\gamma\beta\cos\theta} = p'$$

$$\Rightarrow \quad l'x + m'y + n'\gamma z = p'(\gamma - \gamma\beta z) + n'\gamma\beta$$

$$\Rightarrow \quad l'x + m'y + n'z = \gamma p' - \gamma\beta p' z + n'\gamma\beta$$

$$\Rightarrow \quad l'x + m'y + (n' + \gamma\beta p')z = \gamma(p' + n'\beta)$$

or $\underbrace{lx + my + nz = p}_{\text{Eq}^{\underline{n}} \text{ of plane in S}}$, where $l = l'$, $m = m'$, $n = n' + \gamma\beta p'$ & $p = \gamma(p' + n'\beta)$