Today: Section 16.1
Next time: Section 16.2

We can take a rigid body as a special case of "a system of particles" that we studied previously.

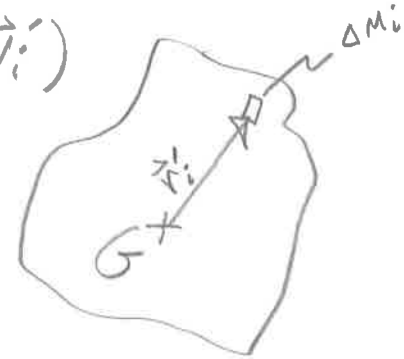
In chapter 14 we saw that for a system of particles

Forces: $\Sigma \vec{F} = m\vec{\bar{a}}$, where the bar over acceleration represents center-of-mass

Moments (torques): $\Sigma \vec{M}_G = \dot{\vec{H}}_G$, where G refers to the center-of-mass coordinate system.

<u>Note:</u> We are going to take the rigid body as being made up of an ∞ number of point particles.

We can write $\vec{H}_G = \sum_i (\vec{r}_i' \times \Delta m_i \vec{v}_i')$



For plane motion $\vec{r}_i' \times \vec{v}_i' = r_i' v_i'$ (Δm_i can only rotate about G) with direction of $\vec{\omega}$

Also $\vec{v}_i' = \vec{\omega} \times \vec{r}_i'$, so $\vec{H}_G = \vec{\omega}\left[\sum_i |\vec{r}_i'|^2 \Delta m_i\right]$

Define $\bar{I} \equiv$ moment of inertia about G

& $\bar{I} \equiv \sum_i |\vec{r}_i'|^2 \Delta m_i$

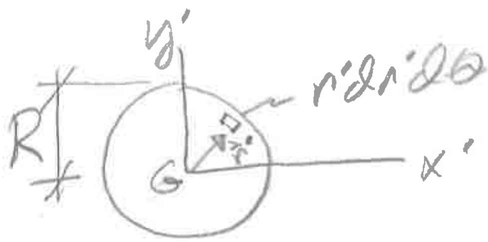
so that $\vec{H}_G = \bar{I}\vec{\omega} \Rightarrow \dot{\vec{H}}_G = \bar{I}\vec{\alpha}$

$I \equiv \sum_i |\vec{r}_i'|^2 \Delta m_i$. If mass is uniform over 2-d surface with area A, then $\Delta m_i = \left(\frac{M_{TOT}}{A}\right) \Delta x_i' \Delta y_i'$

Now $I = \sum |\vec{r}_i'|^2 \left(\frac{M_{TOT}}{A}\right) \Delta x' \Delta y' \doteq \lim_{\substack{\Delta x' \to 0 \\ \Delta y' \to 0}} \sum_i |\vec{r}_i'|^2 \left(\frac{M_{TOT}}{A}\right) \Delta x' \Delta y'$

$= \iint r'^2 \frac{M_{TOT}}{A} dx\, dy$. For polar coordinates $dx'dy' \to r'dr'd\theta$

Example: Uniform disk of radius R



$$I = \left(\frac{M_{TOT}}{A}\right) \int_0^R \int_0^{2\pi} (r')^2 r' dr' d\theta'$$

note: $A = \pi R^2$ so $I = \left(\frac{M_{TOT}\, 2\pi}{\pi R^2}\right) \int_0^R (r')^3 dr'$

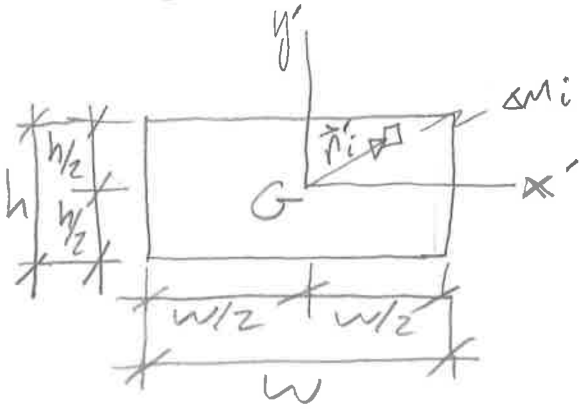
$\Rightarrow \quad I = \left(\frac{2M_{TOT}}{R^2}\right)\left(\frac{R^4}{4}\right) \Rightarrow \quad I = \frac{m_{TOT} R^2}{2}$

Or simply $I = mR^2/2$, where m is understood to be M_{TOT}

Now, angular momentum about G:

$\vec{H}_G = I \vec{\omega} = \frac{MR^2}{2} \vec{\omega}$

Uniform Plate:



$$\bar{I} = \frac{m}{A}\iint (r')^2\,dx'dy'$$

$$= \frac{m}{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{w}{2}}^{\frac{w}{2}} (x'^2 + y'^2)\,dx'dy'$$

$$\Rightarrow \bar{I} = \frac{m}{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{x'^3}{3} + x'y'^2\right)\Big|_{-w/2}^{w/2} dy'$$

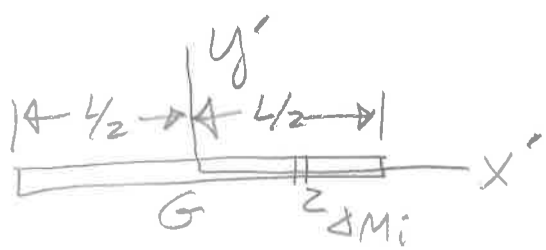
$$\Rightarrow \bar{I} = \frac{m}{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\left(\frac{w^3}{8*3} + \frac{w}{2}y'\right) - \left(-\frac{w^3}{8*3} - \frac{w}{2}y'\right)\right] dy' = \left(\frac{m}{A}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{w^3}{12} + wy'^2\right) dy'$$

$$\Rightarrow \bar{I} = \left(\frac{m}{A}\right)\left(\frac{w^3}{12}y' + \frac{wy'^3}{3}\right)\Big|_{-h/2}^{h/2} = \left(\frac{m}{A}\right)\left(\frac{w^3h}{12} + \frac{wh^3}{12}\right)$$

But $A = wh$ so $\bar{I} = \left(\frac{m}{12wh}\right)(w^3h + wh^3)$

$$\Rightarrow \bar{I} = \left(\frac{m}{12}\right)(w^2 + h^2) \Rightarrow \vec{H}_G = \bar{I}\vec{\omega} = \left(\frac{m}{12}\right)(w^2 + h^2)\vec{\omega}$$

Slender rod of length L:



This is the same as the uniform plate with $w \to L$ & $h \to 0$

so $\bar{I} = \frac{m}{12}L^2 \Rightarrow \vec{H}_G = \bar{I}\vec{\omega} = \frac{m}{12}L^2\vec{\omega}$

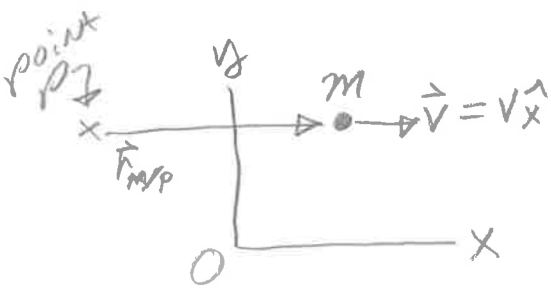
From Book: "The system of the external forces and moments is equipollent to the system consisting of the vector $m\bar{\vec{a}}$ attached to G and the couple of moment $\dot{\vec{H}}_G$."

For plane motion of a rigid body we have $\Sigma F_x = m\bar{a}_x$, $\Sigma F_y = m\bar{a}_y$ & $\Sigma \vec{M}_G = \bar{I}\vec{\alpha}$

From Book: "The mass center G of a rigid body in plane motion moves as if the entire mass of the body were concentrated at that point, and as if all the external forces act on it."

What if we want to sum moments about a point other than G?

First we will look at a point particle



Linear momentum: $\vec{L} = mV\hat{x}$

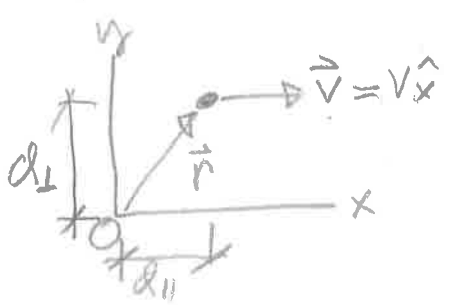
Angular momentum about point p:

$\vec{H}_p = \vec{r}_{M/p} \times \vec{L} = (r_{M/p}\hat{x}) \times (mV\hat{x}) = 0$

$\Rightarrow$ No angular momentum (as expected).

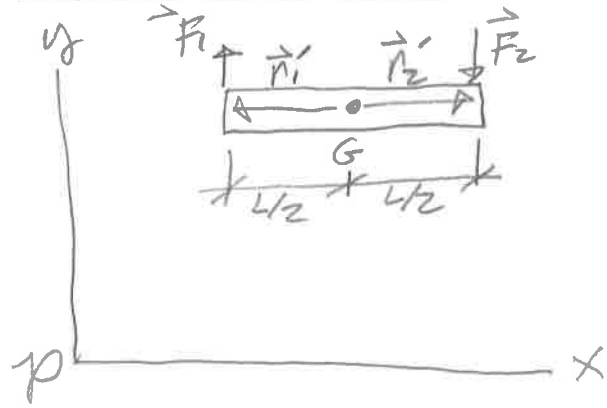
Angular momentum about point O:

$\vec{H}_O = \vec{r} \times \vec{L} = (d_{\parallel}\hat{x} + d_{\perp}\hat{y}) \times (mV\hat{x}) = d_{\perp}mV(-\hat{z})$



In this case $\vec{H}_O$ is not zero

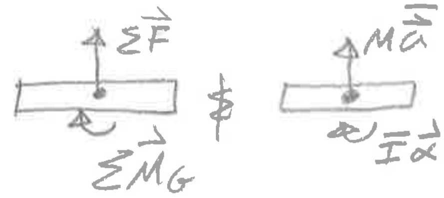
Slender rod with some forces



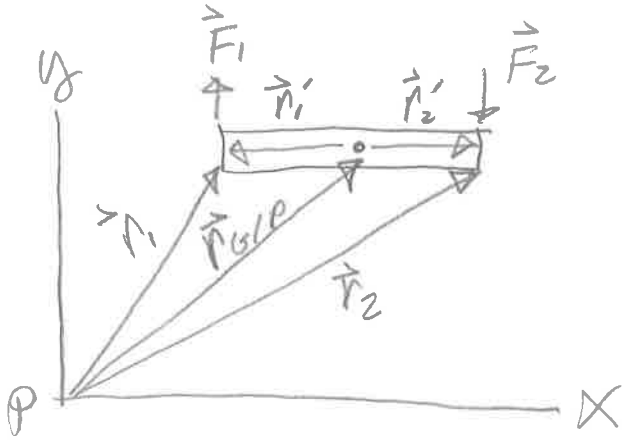
$$\Sigma \vec{F} = m\vec{\bar{a}}$$

$$\text{\&}\ \Sigma \vec{M}_G = \vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2 = \bar{I}\vec{\alpha}$$

Same as



What if we want moments about point P?



<u>Note:</u> $\vec{r}_1 = \vec{r}_{G/P} + \vec{r}_1'$

& $\vec{r}_2 = \vec{r}_{G/P} + \vec{r}_2'$

Now $\Sigma\vec{M}_P = [\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2]$

$$\Rightarrow \Sigma\vec{M}_P = \vec{r}_{G/P} \times \vec{F}_1 + \vec{r}_1' \times \vec{F}_1 + \vec{r}_{G/P} \times \vec{F}_2 + \vec{r}_2' \times \vec{F}_2$$

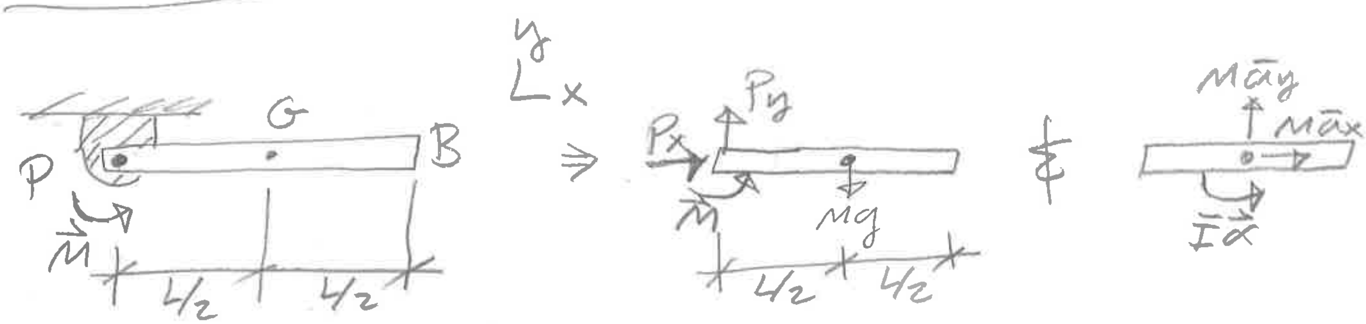
$$\Rightarrow \quad \Sigma\vec{M}_P = (\vec{r}_{G/P}) \times (\Sigma\vec{F}) + \underbrace{[\vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2]}_{\text{Same as } \Sigma\vec{M}_G}$$

So $\Sigma\vec{M}_P = (\Sigma\vec{M}_G) + (\vec{r}_{G/P}) \times (\Sigma\vec{F})$

But $\Sigma\vec{F} = m\vec{\bar{a}}$ & $\Sigma\vec{M}_G = \bar{I}\vec{\alpha}$

$$\Rightarrow \quad \Sigma\vec{M}_P = \bar{I}\vec{\alpha} + \underbrace{\vec{r}_{G/P} \times m\vec{\bar{a}}}_{\text{New piece}}$$

Example:



About G: $\circlearrowleft \sum M_G = \bar{I}\vec{\alpha} \Rightarrow M - \frac{L}{2}P_y = \bar{I}\alpha$ (1)

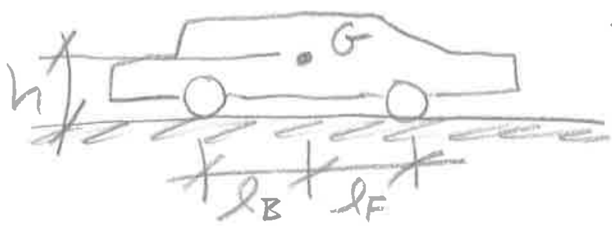
About P: $\circlearrowleft \sum M_P = \bar{I}\vec{\alpha} + \vec{r}_{G/P} \times m\vec{\bar{a}} \Rightarrow M - \frac{L}{2}G = \bar{I}\alpha + \frac{L}{2}m\bar{a}_y$ (2)

But $\sum F_y = m\bar{a}_y \Rightarrow P_y - mg = m\bar{a}_y$ (3)

Eqn 3 into Eqn 2 $\Rightarrow M - \frac{L}{2}mg = \bar{I}\alpha + \left(\frac{L}{2}\right)(P_y - mg)$

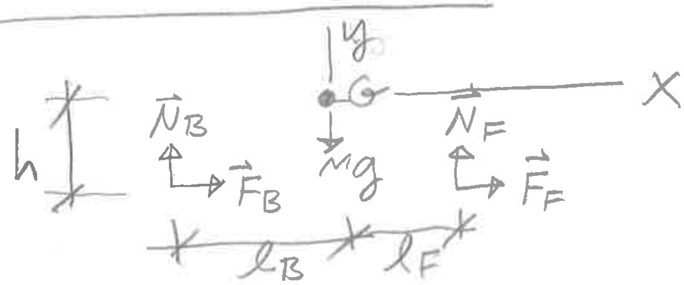
$\Rightarrow$ $M - \frac{L}{2}P_y = \bar{I}\alpha$: Same as Eqn 1 (as it should)

Notes on 16.3: $h = 20\text{in}$, $\ell_B = 60\text{in}$, $\ell_F = 40\text{in}$, $\mu_s = 0.8$



Find max acceleration for
(a) Four wheel drive
(b) Rear wheel drive
(c) Front wheel drive

Four wheel drive:



$\sum F_y = 0 \Rightarrow N_B + N_F = mg$

$\sum F_x = N_B\mu_s + N_F\mu_s = ma_x$

Just need to solve for a_x

$\Rightarrow (N_B + N_F)\mu_s = ma_x$

No need for moment analysis ☺

continued →

Rear wheel drive: In this case

$\vec{F}_F = 0$ & $\vec{F}_B \neq 0$

Front wheel drive: Similar to rear wheel drive case but with $\vec{F}_F \neq 0$ & $\vec{F}_B = 0$

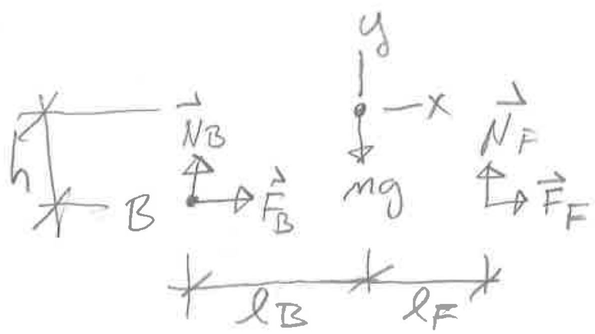
Here $\Sigma F_x = m a_x \Rightarrow N_F \mu_s = m a_x$ (1)

& $\Sigma F_y = 0 \Rightarrow N_B + N_F = mg$ (2) as before

Now need moment analysis.

Need to choose reference point for moment analysis. I am choosing point B (where rear wheels touch road)

$$\curvearrowleft \Sigma M_B = \bar{I}\vec{\alpha} + \vec{r}_{G/B} \times m\vec{a}$$



here $\vec{\alpha} = 0$ &

$$\vec{r}_{G/B} \times m\vec{a} = (l_B\hat{x} + h\hat{y}) \times m a_x \hat{x}$$

$$= h m a_x (-\hat{z}) = h m a_x \curvearrowright$$

so $\curvearrowleft \Sigma M_B = \bar{I}\vec{\alpha} + \vec{r}_{G/B} \times m\vec{a}$

But Eqn 1 $\Rightarrow N_F = \frac{m a_x}{\mu_s}$

$$\Rightarrow l_B m g - (l_B + l_F) N_F = h m a_x$$

$$\Rightarrow l_B m g = h m a_x + (l_B + l_F) m a_x / \mu_s$$

$$\Rightarrow a_x = \left[\frac{l_B g}{h + (l_B + l_F)/\mu_s}\right] = \left[\frac{60 * 32.2}{20 + 100/0.8}\right] \frac{ft}{s^2} = 13.32\ ft/s^2$$