Today: Section 15.3 and 15.4

Next time: Section 16.1

Instantaneous center of rotation:

Take a rigid body that is in plane motion with a rotation $\vec{\omega}_A$ about some point A that has velocity $\vec{v}_A$

For point A we could find some other point C such that $\vec{v}_A = \vec{\omega}_A \times \vec{r}_{A/C}$

Note: since $\vec{r}_{A/C} \cdot \vec{v}_A = \vec{r}_{A/C} \cdot (\vec{\omega}_A \times \vec{r}_{A/C}) = 0$, $\vec{r}_{A/C}$ & $\vec{v}_A$ must be perpendicular.

What about some other point on the rigid body?

Take some point B on the rigid body.

We have $\vec{v}_A = \vec{\omega}_A \times \vec{r}_{A/C}$ & want to construct $\vec{v}_B$: $\vec{v}_B = \vec{v}_{B/A} + \vec{v}_A$

$$= \vec{\omega}_A \times \vec{r}_{B/A} + \vec{\omega}_A \times \vec{r}_{A/C} = \vec{\omega}_A \times (\vec{r}_{B/A} + \vec{r}_{A/C})$$

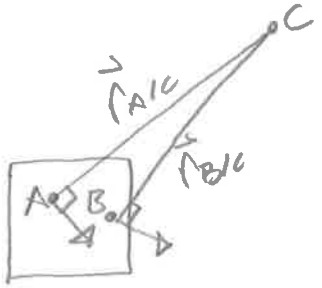
$$= \vec{\omega}_A \times (\vec{r}_B - \vec{r}_A + \vec{r}_A - \vec{r}_C) = \vec{\omega}_A \times (\vec{r}_B - \vec{r}_C)$$

$$= \vec{\omega}_A \times \vec{r}_{B/C}$$

We now have $\vec{v}_B$ constructed as a rotation about point C. Since B was arbitrary, this works for all points on the rigid body!

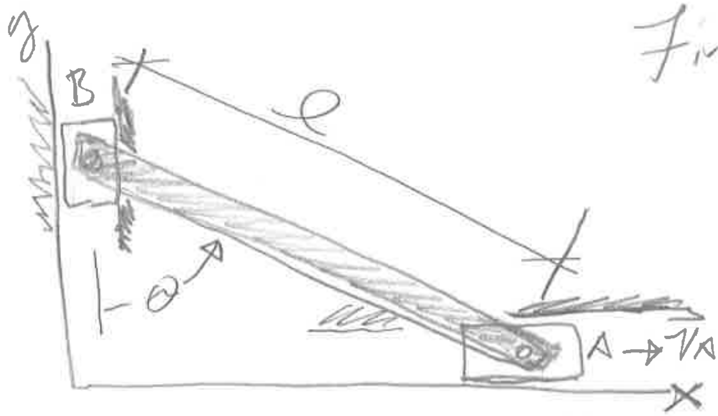
<u>For this one instance in time</u> we have a single center of rotation for all points on the rigid body.

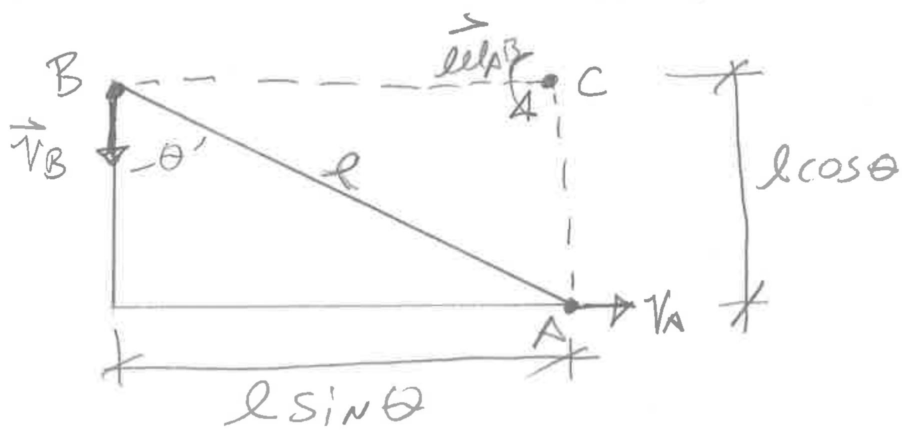
As with point A, the velocity of point B is perpendicular to the line connecting it to point C



Example: Given $\vec{V}_A = V_A \hat{x}$, l & θ

Find $\vec{V}_B$ & $\vec{\omega}_{AB}$





$$V_A = \omega_{AB}\, l\cos\theta$$

$$\Rightarrow \vec{\omega}_{AB} = \left(\frac{V_A}{l\cos\theta}\right) \circlearrowleft$$

$$\& \quad \vec{V}_B = \omega_{AB}\, l\sin\theta \downarrow$$

$$= \left(\frac{V_A}{l\cos\theta}\right) l\sin\theta \downarrow$$

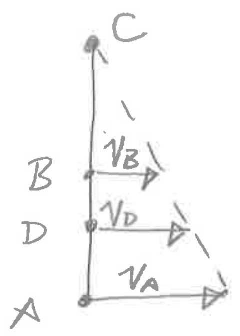
$$= V_A \tan\theta \downarrow$$

This method for this problem is easier to solve than our previous method.

For every point P on the rigid body (in plane motion that is rotating) $v_P = \omega r_{P/C}$

Here ω is fixed $\Rightarrow$ v_P scales with $r_{P/C}$

$\Rightarrow$

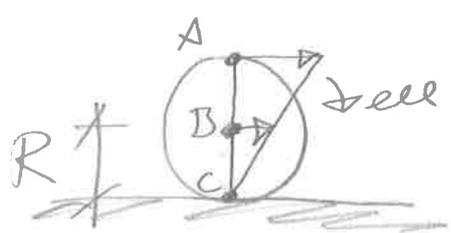


If point B is ½ distance from C as is A, then $v_B = \frac{1}{2} v_A$.
If point D is 3/4 distance from C as is A, then $v_D = \frac{3}{4} v_A$
And on & on.

This forms similar triangles $\Rightarrow$ $\frac{v_B}{r_{B/C}} = \frac{v_D}{r_{D/C}} = \frac{v_A}{r_{A/C}}$

Wheel rolling w/o slipping: we know that

$v_A = 2 v_B$ & $v_C = 0$



In fact, we now know that point C is the instantaneous center of rotation.

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§15.4 Relative motion for two points:

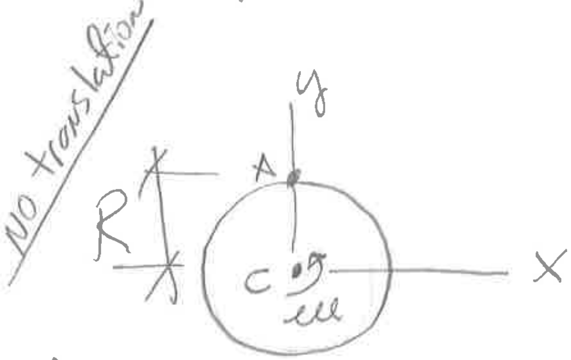
$\vec{r}_{A/B} \equiv \vec{r}_A - \vec{r}_B \Rightarrow \vec{r}_A = \vec{r}_{A/B} + \vec{r}_B$ & $\vec{v} \equiv \frac{d\vec{r}}{dt} \Rightarrow \vec{v}_A = \vec{v}_{A/B} + \vec{v}_B$

For a rigid body, the distance between A & B is fixed $\Rightarrow$ $\vec{v}_{A/B}$ is purely rotational so $\vec{v}_{A/B} = \vec{\omega} \times \vec{r}_{A/B}$

<u>Acceleration</u> $\vec{a} \equiv \frac{d\vec{v}}{dt} \Rightarrow \vec{a}_A = \vec{a}_{A/B} + \vec{a}_B$ where

$\vec{a}_{A/B} = \vec{\alpha} \times \vec{r}_{A/B} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{A/B}]$

In the past we used $\vec{a} = a_N \hat{e}_N + a_t \hat{e}_t$ for circular motion. How does this relate to our new expression?



$\vec{\omega} = \omega \hat{z}$ with ω = constant

$\vec{a}_A = a_N \hat{e}_N + a_t \hat{e}_t$, where

$a_N = \frac{V_A^2}{\rho}$ & $a_t = \frac{dV_A}{dt}$

$\hat{e}_N$ points from A to C and $\hat{e}_t$ is tangential to the path that A takes. For uniform circular motion $V_A = R\omega$ & $\frac{dV_A}{dt} = 0$ so

$a_N = R\omega^2$ & $a_t = 0$ & $\hat{e}_N = -\hat{y}$ so

$\vec{a}_{A/C} = \vec{a}_A = R\omega^2(-\hat{y})$

But if $\omega \neq$ constant we get

$\frac{dV}{dt} = R\frac{d\omega}{dt} = R\alpha$ and $\hat{e}_t = -\hat{x}$ now

$\vec{a}_{A/C} = \vec{a}_A = \alpha R(-\hat{x}) + R\omega^2(-\hat{y})$

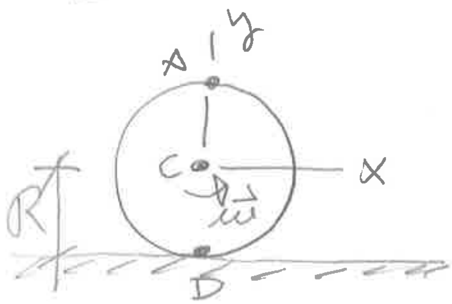
Our expression $\vec{a}_{A/C} = \alpha R(-\hat{x}) + R\omega^2(-\hat{y})$ has to match what we would get using

$\vec{a}_{A/C} = \vec{\alpha} \times \vec{r}_{A/C} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{A/C}]$. Here $\vec{\alpha} = \alpha \hat{z}$ & $\vec{\omega} = \omega \hat{z}$ & $\vec{r}_{A/C} = R\hat{y}$ so $\vec{a}_{A/C} = \alpha R(\hat{z} \times \hat{y}) + \omega^2 R\, \hat{z} \times [\hat{z} \times \hat{y}]$

But $\hat{z} \times \hat{y} = -\hat{x}$ & $\hat{z} \times (-\hat{x}) = -\hat{y}$ so $\vec{a}_{A/C} = \alpha R(-\hat{x}) + \omega^2 R \hat{z} \times (-\hat{x})$

$\Rightarrow \vec{a}_{A/C} = \alpha R(-\hat{x}) + \omega^2 R(-\hat{y})$ This matches previous expression ☺

Example: Wheel rolling with no slipping



$\vec{a}_A = \vec{a}_{A/C} + \vec{a}_C$ If $\vec{V}_C = \text{const.}$

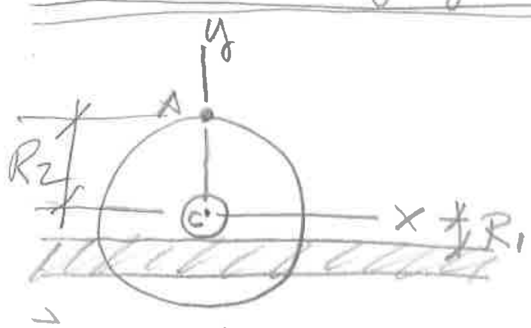
then $\vec{a}_C = \vec{0}$ & $\vec{\alpha} = \vec{0}$ & $V_C = R\omega$

so $\vec{a}_A = R\omega^2(-\hat{y})$

If $\vec{V}_C \neq \text{const}$ then $\vec{\alpha} \neq \vec{0}$ & $\vec{a}_C \neq 0$

$$\Rightarrow \quad \vec{a}_A = \alpha R(-\hat{x}) + \omega^2 R(-\hat{y}) + \vec{a}_C$$

Example: Flywheel on shaft rolling on rail



Given V_C, a_C, R_1 & R_2 find $\vec{a}_A$:

$\vec{a}_A = \vec{a}_{A/C} + \vec{a}_C$

where $\vec{a}_{A/C} = \vec{\alpha} \times \vec{r}_{A/C} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{A/C}]$

$\vec{V}_C = V_C\hat{x} = R_1\omega\,\hat{x}$ so $\vec{\omega} = (V_C/R_1)(-\hat{z})$

& $\vec{a}_C = a_C\hat{x} = R_1\alpha\,\hat{x}$ so $\vec{\alpha} = (a_C/R_1)(-\hat{z})$

Note: Assumed wheel is moving in +x-direction.

If wheel is moving in negative x-direction, then $V_C \to$ negative and sign of $\vec{\omega}$ is still good

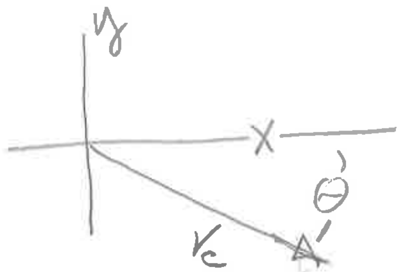
We also have $\vec{r}_{A/C} = R_2\hat{y}$ so $\vec{a}_A = \vec{a}_{A/C} + \vec{a}_C$

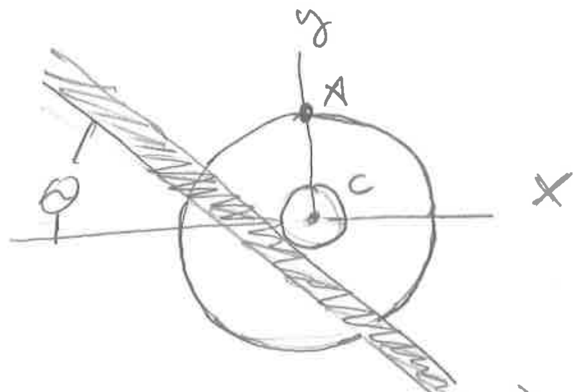
$$\Rightarrow \quad \vec{a}_A = \frac{a_C}{R_1}R_2(-\hat{z})\times\hat{y} + \frac{V_C^2}{R_1^2}R_2\,\hat{z}\times[\hat{z}\times\hat{y}] + \vec{a}_C$$

$$\Rightarrow \quad \vec{a}_A = \left(\frac{a_C R_2}{R_1}\right)\hat{x} + a_C\hat{x} + \left(\frac{V_C^2 R_2}{R_1^2}\right)(-\hat{y})$$

Note: In this case $\vec{a}_c = a_c\hat{x}$

If rail was tilted some angle θ with respect to horizontal then





$\vec{V}_c = V_c\cos\theta\,\hat{x} - V_c\sin\theta\,\hat{y}$ (taking θ as positive)

Note: if we took θ as negative then

$\vec{V}_c = V_c\cos\theta\,\hat{x} + V_c\sin\theta\,\hat{y}$ Just have to be consistent with sign convention.

Also (keeping $\theta > 0$)

$\vec{a}_c = a_c\cos\theta\,\hat{x} - a_c\sin\theta\,\hat{y}$

now

$$\vec{a}_A = \left[\left(\frac{a_c R_2}{R_1}\right)\hat{x} + a_c\cos\theta\,\hat{x}\right] + \left[\left(\frac{V_c^2}{R_1^2}R_2\right)(-\hat{y}) + a_c\sin\theta\,\hat{y}\right]$$